Social Lending*

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ABSTRACT

Prosper, the largest online social lending marketplace with nearly a million members and \$178 million in funded loans, uses an auction amongst lenders to finance each loan. In each auction, the borrower specifies D, the amount he wants to borrow, and a maximum acceptable interest rate R. Lenders specify the amounts a_i they want to lend, and bid on the interest rate, b_i, they're willing to receive. Given that a basic premise of social lending is cheap loans for borrowers, how does the Prosper auction do in terms of the borrower's payment, when lenders are *strategic agents* with private true interest rates?

The Prosper mechanism is exactly the same as the VCG mechanism applied to a *modified instance* of the problem, where lender i is replaced by a_i dummy lenders, each willing to lend one unit at interest rate b_i. However, the two mechanisms behave very differently — the VCG mechanism is truthful, whereas Prosper is not, and the total payment of the borrower can be vastly different in the two mechanisms. We first provide a complete analysis and characterization of the Nash equilibria of the Prosper mechanism. Next, we show that while the borrower's payment in the VCG mechanism is *always* within a factor of $O(\log D)$ of the payment in any equilibrium of Prosper, even the cheapest Nash equilibrium of the Prosper mechanism can be as large as a factor D of the VCG payment; both factors are tight. Thus, while the Prosper mechanism is a simple uniform price mechanism, it can lead to much larger payments for the borrower than the VCG mechanism. Finally, we provide a model to study Prosper as a dynamic auction, and give tight bounds on the price for a general class of bidding strategies.

1. INTRODUCTION

Social lending, or peer-to-peer lending, is an emerging alternative to banks and personal loans, allowing individuals to lend/borrow money to each other directly without the participation of traditional financial intermediaries. The Internet has contributed significantly to the feasibility and growth of social lending, by providing the technology for an online marketplace to match lenders and borrowers. Social lending offers borrowers the opportunity to obtain loans at lower interest rates and costs, and lenders with an opportunity for investments with higher rates of return than from banks or other common alternatives. In this paper, we study the auction mechanism used to finance loans by the largest such social lending marketplace.

Social lending is now a large business on the Internet, with an increasing number of companies such as Prosper (US) [2], Zopa (UK) [3], and Lending Club [1], to name a few, offering online social lending services. The total amount of money borrowed using such peer-to-peer loans was approximately \$650 million in 2007, and is projected to reach \$5.8 *billion* in 2010. The number of users (borrowers and lenders) on social lending websites is also staggering, with more than one million members on Prosper and Zopa alone. Given the large volume of trade, evidenced by the large number of users and the vast sums of money being lent and borrowed, social lending is clearly a significant component of electronic commerce.

The largest social lending site in the US is Prosper, with over $830,000$ members, and over \$178 million in funded loans. Prosper, which describes itself as an "eBay for loans", auctions off loans amongst interested lenders, using competition amongst lenders to bring down the final interest rate for the borrower. Borrowers create loan listings, specifying the amount of money they are willing to borrow and a reserve interest rate, the maximum rate that they are willing to accept. Lenders choose individual listings to bid on, specifying an amount and interest rate for each loan. In addition to standard criteria such as credit scores and histories, lenders can also consider a borrower's personal story, endorsements from friends, and group affiliations. The bidding starts at the reserve rate, and lenders can then bid down the interest rate in an auction. When the auction ends, Prosper combines the bids with the lowest interest rates into a single loan to the borrower and handles all loan administration tasks including loan repayment and collections on behalf of the matched borrower and lenders.

A basic premise of social lending is cheap loans for borrowers. To what extent does the mechanism used to fund loans lead to small borrower payments, *given that lenders act strategically, as selfish and rational agents*? Each lender has a private interest rate, which is the minimum interest

*NOTE: **This submission is exactly 10 pages. For want of space, some proofs are only included as supplementary material in the Appendix.**

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rate at which she is willing to invest in a particular loan. While the Prosper mechanism certainly selects the lenders with the lowest interest rates to finance the loan, this does not necessarily lead to a cheap loan — the rate reported by a lender need not be her true private interest rate, since reporting a higher interest rate might lead to a better return. (That lenders are strategic is clearly evidenced by their behavior in Prosper: lenders are allowed to, and indeed do, decrease their rates to increase their allocation through the course of the auction.) Given that lenders behave strategically, how does the choice of a mechanism — that consists of an allocation and payment rule — affect the total payment of the borrower?

The auction used by Prosper, which is the primary focus of this paper, turns out to be a fairly curious mechanism. It is VCG-like, but not quite VCG — it is, in fact, a uniform price mechanism obtained by applying VCG to a *modified instance* of the problem, as described below. Suppose the borrower wants to borrow an amount D, and each lender i specifies her budget a_i and her offered interest rate b_i. Replace every lender i by a_i dummy lenders with budget 1 and interest rate b_i each. Now run VCG on this new instance to determine the winners and their payments. Recall that the VCG mechanism to buy k identical items from competing sellers (here, $k = D$ and each item is a unit of money) buys from the k cheapest sellers and pays each of them the same price, which is the bid of the $(k+1)$-*th* lowest bid. Thus, applying VCG to the modified instance yields a solution where all winning lenders receive the same interest rate, which is either the bid of the first loser or the last winner, depending on whether or not the last winner exhausts her budget. Of course, the VCG mechanism can also be applied directly to the input (a_i, b_i) without modification, yielding an incentive compatible mechanism.

1.1 Overview of Results

We first provide a complete analysis of the Nash equilibria of the Prosper mechanism modeled as a one-shot auction game of complete information (§3). Since the Prosper mechanism is a uniform price mechanism (*i.e.*, every winner receives the same interest rate, also called the *price*), and we are interested in the borrower's payment, we focus on the set of possible prices that can arise in a Nash equilibrium. We first show that computing the Nash equilibria with the smallest and largest prices (or equivalently, total payments) is, in general, NP-hard, and hard to approximate within any reasonable factor. However, this hardness vanishes when losers are restricted to bid their true interest rate[1] — in this case, we show how to completely characterize the equilibria of the Prosper mechanism. Finally, as we show in §3.2, this characterization can be sharpened further if we restrict ourselves to equilibria where winners do not bid less than their true interest rates.

Next we compare the Prosper mechanism against the VCG mechanism from the perspective of the borrower's payment (§4). As the Prosper mechanism is not incentive compatible, we compare the payment in the *best* and *worst* Nash equilibria of the Prosper mechanism against that of the VCG outcome. While no mechanism dominates the other, the VCG mechanism leads to a payment that is always within a factor of $O(\log D)$ of the cheapest Nash equilibrium of the Prosper mechanism, whereas even the cheapest Nash equilibrium of the Prosper mechanism can be as large as a factor D of the VCG payment (both factors are tight). A similar result holds for the worst Nash equilibrium of the Prosper mechanism.

In §5, we investigate two other natural uniform price mechanisms that are closely related to Prosper, and compare their equilibria and payments. Finally in §6, we examine the Prosper mechanism when modeled as a dynamic auction, and provide tight bounds on the price for a general class of bidding strategies.

1.2 Related Work

While social lending is a large and growing aspect of commerce on the Internet, it has received only limited attention in the research literature. The most relevant work is that of Freedman and Jin [9], where the authors examine the functioning of Prosper based on transaction data. They establish relations between interest rates, actual returns, default rates, and credit grades, and compare them with loans contracted in traditional banks. However, their work is entirely empirical and does not attempt to model or analyze the mechanisms of social lending from a theoretical standpoint.

The auction used in Prosper is a particular instance of a uniform price reverse auction with multi-unit demand. While multi-unit auctions have been well-studied in the economics literature (see, for example, Krishna [13]), they differ from our work in two ways. First, most work on uniform price auctions focuses on the unit-demand setting, where each bidder is interested in only one item (this case corresponds to the VCG mechanism as we note above). In contrast, bidders in Prosper generally have multi-unit demand (or rather supply, since Prosper is a reverse auction). The fact that bidders desire more than one item makes the incentives very different compared to those of the single-unit demand setting. Uniform price auctions for multi-unit demand have been studied as well, but either the mechanism used is different from Prosper, or the setting is different (for instance, Draaisma and Noussair [**?**] studied a uniform price auction where bidders' demands are restricted to at most two units); also, equilibrium characterizations are usually very partial and complex.

Second, the economics literature has typically focused on characterizing Bayes-Nash equilibria whereas in this paper, we will focus on Nash equilibria. Indeed, the Prosper mechanism is dynamic, in the sense that lenders can get information on other lender's bids and modify their own bids through the course of the auction. In such cases, it is preferable to model the auction as a one-shot game of complete information, as Varian observes for the similar situation of auctions for online advertising [15]. This is in part motivated by the fact that, even with minimal knowledge about one's opponent, strategies of rational players in a repeated game converge to a Nash equilibrium of the one-shot game (Kalai and Lehrer [11]). Also, we will be particularly interested in *worst-case* comparisons between the borrower's payment in different mechanisms, an approach more specific to the computer science literature.

Finally, our work is similar in spirit to the literature on *frugal* mechanism design for hiring a team problems, which studies mechanisms with small payments for the buyer in a reverse auction. In the *hiring a team* problem [4, 14, 8, 7,

[1]Note that the true interest rate is indeed the lowest *safe* bid for a loser if other lenders deviate unexpectedly from an equilibrium.

10, 12, 6], a principal wants to hire a team of selfish agents at a low cost to perform a task, each agent having a private cost for performing her sub-task. Only feasible teams are able to complete the task. In the context of social lending, one can consider lenders as agents, and a feasible team is simply one whose total budget is greater than or equal to the borrower's demand. However, our work differs significantly from the *hiring a team* literature: the system of feasible sets in our setting is quite different from that considered in the frugality literature and the existing results do not apply to the feasible sets of our social lending setting. Second, we do not attempt to derive the optimal incentive compatible frugal mechanism, but rather examine the most commonly used social lending mechanism, and compare it with other natural alternatives.

To the best of our knowledge, this is the first paper to study the auction mechanisms used in social lending.

2. PRELIMINARIES

In this paper, we focus on a single auction corresponding to a single loan. The borrower wants to borrow an amount of money D, hereafter referred to as the *demand*. He also has a reserve interest rate R, which is the maximum interest rate he is willing to pay for the loan. Multiple lenders, denoted by $L_1, \ldots, L_n$, compete to finance the loan. Each lender L_i specifies the amount a_i she's willing to lend (referred to as her *budget*), and her *bid* b_i, which is the interest rate she seeks from her loan. The demand D and the budgets a_i are integers, *i.e.*, they are expressed in cents (or the smallest unit of currency). We assume that the budgets a_i are public information, *i.e.*, lenders do not behave strategically with respect to their budgets, and that R and D are publicly known, *i.e.*, the borrower is not strategic. Finally, we assume that there is no lender with monopoly; that is, for any j, $\sum_{i \neq j} a_i \geq D$.

A *mechanism* for this setting computes an *allocation* and "*price*" for each lender, given the lenders' budgets $a_1, \ldots, a_n$, and bids $b_1, \ldots, b_n$. The allocation for lender L_i, $0 \leq x_i \leq a_i$, is the amount borrowed from lender L_i, and the price[2] p_i is the effective interest rate[3] at which the lender will be paid back by the borrower. In our analysis we require that the total allocation exactly funds the loan, that is, $\sum_i x_i = D$ (note that the no-monopoly assumption guarantees that this is possible). Also, to ensure voluntary participation, we will require that $p_i \geq b_i$. We say that lender L_i is a *winner* if she receives a positive allocation $x_i > 0$.

We suppose that every lender L_i has a private true interest rate r_i. This could be, for instance, the rate of return that the lender expects to get if she chooses an alternative investment option. Other factors such as the risk associated with a particular loan (that is, the risk of the borrower defaulting on the loan) could also affect the true rate r_i. However, our purpose is not to determine how the lender decides on his private interest rate; rather, we simply take it as given.

Lenders are rational, which means they act to maximize their *utility* given their true interest rates. The utility of lender L_i is defined as

$$u_i = x_i(p_i - r_i) \ .$$

[2]Note the distinction between the payment and the price: the payment is the product of the allocation and the price.
[3]The effective interest rate is the ratio of the payment to the allocation.

All the mechanisms we consider in this paper will use the following allocation rule.

Definition 2.1. (Allocation $\mathcal{A}(\mathbf{b})$, Last Winner and First Loser). *Given a bid profile* $\mathbf{b} = (b_1, \ldots, b_n)$, *order lenders so that* $b_1 \leq b_2 \leq \cdots \leq b_n$. *Let* $k = \min\{j \mid \sum_{i=1}^{j} a_i \geq D, j = 1, \ldots, n\}$. *Then the allocation* $\mathcal{A}(\mathbf{b})$ *is defined as* $x_i = a_i$ *for* $i < k$, $x_k = D - \sum_{i=1}^{k-1} a_i$, *and* $x_i = 0$ *for* $i > k$. *We refer to* L_k *as the* last winner *and* L_{k+1} *as the* first loser.

Note that there is *at most one lender* — the last winner — who might not exhaust her budget.

Throughout the paper we will use k as index for the last winner and $k+1$ as index for the first loser. Note also that the ordering and index of lenders can change from one bid profile to another.

When multiple lenders bid the same interest rates, a fixed, preannounced tie-breaking rule is necessary. To maximize clarity of presentation, we use a tie-breaking rule which has oracle access to lenders' true interest rates. (Ties between lenders with the same true interest rate are broken arbitrarily.) While oracle access might appear to be a very strong assumption, all of the results in our paper hold, up to modification by ϵ, for *any fixed preannounced tie-breaking rule*, as explained in the footnote below[4].

For completeness, we recall the definition of a Nash equilibrium.

Definition 2.2 (Nash equilibrium). *A bid profile* $\mathbf{b} = (b_1, \ldots, b_n)$ *is a Nash equilibrium if no lender can increase her utility by unilaterally changing her bid, that is, keeping the bids of other lenders fixed.*

Given a set of bids from lenders, how should one select the winners and decide their respective interest rates? While all the mechanisms investigated in this paper have the same allocation rule, they differ in their payments.

VCG Mechanism. The VCG mechanism is *incentive compatible*, *i.e.*, it is a dominant strategy for every lender to report her true interest rate r_i.

Definition 2.3 (Set Δ and bid profile $\mathbf{r}$). *Define a bid profile* $\mathbf{r} = (r_1, \ldots, r_n)$ *(i.e., everyone bids truthfully). The VCG allocation is computed according to* $\mathcal{A}(\mathbf{r})$ *by Definition 2.1. We denote by* Δ *the set of VCG winners.*

The VCG payments are computed as follows. Let $\Delta(j)$ be the set of winners in VCG after removing lender L_j from the group of lenders, and let $x_i(j)$ be the allocation of each $L_i \in \Delta(j)$. Observe that $\Delta \subset \Delta(j) \cup \{L_j\}$. The net payment to lender L_j in the VCG mechanism is

$$\sum_{L_i \in \Delta(j)} b_i x_i(j) - \sum_{L_i \in \Delta} b_i x_i + b_j x_j \ .$$

[4]Two alternative treatments are to discretize the bidding space to multiples of ϵ (as is actually the case in Prosper, where $\epsilon = 0.05$), or to consider ϵ-Nash equilibria. With the first, every Nash equilibrium with price p (using oracle access) translates to a Nash equilibrium with price p or $p-\epsilon$, depending on the particular tie-breaking rule used; with the second, a Nash equilibrium at price p translates to either a Nash or an ϵ-Nash equilibrium at the same price. See [10, 12] for more discussions on tie-breaking rules.

Note that the VCG mechanism is not a uniform price mechanism. Indeed, the (effective) prices associated with the VCG payments given above are not necessarily the same for all winning lenders.

First Price Auction. Another natural mechanism is the "first price" auction: the allocation is according to $\mathcal{A}(\mathbf{b})$, and each winner is paid his offered interest rate b_i. The first price auction is clearly not incentive compatible; it also need not have a Nash equilibrium. In fact, unlike other settings where the first price mechanism admits an ϵ-Nash equilibrium (such as single item auctions or path auctions [10]), in our setting it need not even have an ϵ-Nash equilibrium, as the following example shows.

EXAMPLE 2.1. *Let $D = 15$ and suppose there are three lenders L_1, L_2, L_3, with budgets $a_1 = a_2 = a_3 = 10$, and interest rate $r_1 = r_2 = 0.1$, $r_3 = 0.5$. Assume ties are broken according to $L_1 > L_2 > L_3$. For any given $\epsilon > 0$, consider the bid b_1 made by L_1. If $b_1 = 0.1$, then L_2 will bid $b_2 = 0.5$ to obtain 5 units of allocation with a total utility of $5 \cdot (0.5 - 0.1) = 2$. If $b_1 = 0.5$, then L_2 will bid $b_2 = 0.5 - \epsilon$, to obtain 10 units of allocation with a total utility of $4 - 10\epsilon$. If $0.1 < b_1 < 0.5$, then L_2 will set either $b_2 = 0.5$ to obtain 5 units of allocation or $b_2 = b_1 - \epsilon$ to obtain 10 units of allocation, whichever utility is larger. Given the interest b_2 set by L_2, lender L_1 will set either $b_1 = 0.5$ or $b_1 = b_2$, whichever utility is larger. Thus, the strategies of L_1 and L_2 form a loop and there is no ϵ-Nash equilibrium.*

Note that the choice of tie-breaking rule is not the reason the first price mechanism does not have an ϵ-Nash equilibrium; it is easy to see that this example does not have an ϵ-Nash equilibrium for any tie-breaking rule.

The Prosper Mechanism. Given a bid profile $\mathbf{b}$, the mechanism used by Prosper, denoted by `PROSPER`, is the following.

- *Allocation*: `PROSPER` computes the allocation according to $\mathcal{A}(\mathbf{b})$ (recall Definition 2.1).

- *Pricing*: If $x_k = a_k$, *i.e.*, the last winner exhausts her budget, the price to each winner is the bid of the first loser, *i.e.* $p_i = b_{k+1}$ for $i = 1, \ldots, k$. We will refer to this interest rate as the *price* throughout. If $x_k < a_k$, *i.e.*, the last winner does not exhaust her budget, the price to all winners is the bid of the last winner, *i.e.* $p_i = b_k$ for $i = 1, \ldots, k$.

`PROSPER` is not truthful for the same reason that the first price auction is not: for example, suppose there are two lenders L_1 and L_2 with $a_1 = D + 1, r_1 = 1$ and $a_2 = D + 1, r_2 = 2$. Then L_1's utility is greater when bidding $b_1 = 2$ than when bidding (truthfully) $b_1 = 1$ (note that ties are broken by $L_1 > L_2$). However, as we will see shortly, `PROSPER` always has a Nash equilibrium, unlike the first price auction.

3. EQUILIBRIUM ANALYSIS

In this section, we will analyze the Nash equilibria of `PROSPER`. We are interested in the borrower's total payment, and `PROSPER` is a uniform price mechanism. We therefore focus on characterizing the set of *prices* that can arise at an equilibrium. We will refer to the equilibrium with the smallest price as the *cheapest* Nash equilibrium, and to that with the largest price as the *worst* Nash equilibrium.

We start by showing that, while we can always construct a Nash equilibrium, computing the cheapest — or worst — equilibrium is hard. This hardness disappears when restricting ourselves to a natural subset of equilibria — that where *losers* bid their true interest rate. We then provide a complete characterization of all equilibria in §3.1. Finally we show in §3.2 that our characterization can be sharpened even further when, in addition, we assume that *winners* bid at least their true rate.

We will assume throughout that losers bid *at least* their true value, *i.e.*, $b_i \geq r_i$ if $x_i = 0$. While our characterization easily extends to equilibria when losers can bid less than their true value[5], we do not include this case since bidding less than the true value is an *unsafe* bidding strategy for losers who obtain zero utility anyway. For instance, suppose a winning lender exits the market, L_i could then become a winner at an interest rate strictly less than her private interest rate, leading to negative utility.

Our first result is that `PROSPER` admits a Nash equilibrium, in contrast to the first price auction.

```
ALG-GREEDY

  1. Define a bid profile b = r = (r_1, ..., r_n).
  2. Repeat the following until convergence:

         If there is L_i ∈ Δ such that L_i can in-
         crease utility by increasing her bid uni-
         laterally in the profile b, increase b_i to
         the utility-maximizing point and update b.

  3. Output the bid profile b.
```

PROPOSITION 3.1. `ALG-GREEDY` *returns a Nash equilibrium of* `PROSPER`.

PROOF. Denote by $\mathbf{b}$ the profile generated by `ALG-GREEDY`. Note that any lender $L_i \in \Delta$ who increases her bid in Step 2 becomes the last winner in the current bid profile. Let L_k denote the last winner in $\mathbf{b}$. Observe that all lenders in Δ are winners and $L_k \in \Delta$, since `ALG-GREEDY` starts with the profile of true rates, and only lenders in Δ can increase their bids to become the last winner.

First, no lender can obtain more utility by increasing her bid in $\mathbf{b}$ — by definition, no winner in Δ wants to increase her bid; if winners not in Δ increase their bid higher than b_k, their allocation falls to zero. On the other hand, all losers bid their true interest (since we start with the profile $\mathbf{b} = \mathbf{r}$) and all winners other than L_k exhaust their budget, so they do not have an incentive to decrease their bid. For L_k, by the rule of `ALG-GREEDY`, L_k is the last lender in Δ who moves her bid up to the point where her utility is maximized. Hence, she cannot obtain more utility by decreasing her bid, and so $\mathbf{b}$ is a Nash equilibrium. ☐

While it is pleasant that `PROSPER` always has a Nash equilibrium, computing the cheapest and worst Nash equilibrium is, in general, NP-hard as the following results show.

[5]Rather than only the prices in Lemma 3.2, we obtain intervals of prices corresponding to each equilibrium price in the restricted setting.

THEOREM 3.1. *The computation of a cheapest Nash equilibrium of* `PROSPER` *is NP-hard. Furthermore, for any polynomial time computable function $f(n)$, it does not admit any approximation algorithm within a ratio of $f(n)$, unless P=NP.*

PROOF. We reduce from Partition: Given an instance of Partition with a set of integers $S = \{x_1, \ldots, x_n\}$ where $\sum_{i=1}^{n} x_i = 2N$, we ask if S can be partitioned into two subsets such that the sum of the numbers in each subset is N. Assume without loss of generality that $1 \le x_i \le N$, for $i = 1, \ldots, n$.

We construct an instance of our problem as follows: Let $M \triangleq f(n)$. For $i = 1, \ldots, n$, there is a lender L_i with budget $a_i = x_i M$ and interest $r_i = 0$. Further, there are two extra lenders L_0 and L_{n+1} with budget $a_0 = MN + 1$, $a_{n+1} = 3MN$, and interest $r_0 = 0$, $r_{n+1} = 1$, respectively. Let $D = 2MN + 1$. We claim that it is NP-hard to distinguish whether the total payment of the cheapest Nash equilibrium is smaller than or equal to $\frac{2MN+1}{MN+1}$ or at least $2MN + 1$.

Assume that there is a partition of S into S_1 and S_2 such that the sum of the numbers in each subset is N. We construct a bid profile $\mathbf{b}$ as follows: Let $b_i = r_i$ for $i = 0, n+1$, $b_i = 0$ if $x_i \in S_1$ and $b_i = \frac{1}{MN+1}$ if $x_i \in S_2$. Given $\mathbf{b}$, as $a_0 + \sum_{i:x_i \in S_1} a_i = 2MN + 1 = D$, the winners are L_0 and those corresponding to the set S_1 and all winners exhaust their budget. Thus, the price to each winner is $\frac{1}{MN+1}$ and the utility of L_0 is $a_0 \cdot \frac{1}{MN+1} = 1$. If L_0 increases her bid b_0 (to a point at most $b_{n+1} = 1$ to remain to be a winner), her payment is at most 1 and utility is at most $1 \cdot (D - \sum_{i:x_i \in S} a_i) = D - 2MN = 1$, which implies that L_0 has no incentive to change her bid. Further, it is easy to see that the lenders corresponding to the set S_1 do not have an incentive to change their bid as well. For each lender L_i, $x_i \in S_2$, although L_i can reduce her bid to 0 to be a winner, the price to winners becomes 0 as well, which leads to a 0 utility to L_i. Therefore, no lender can unilaterally increase her utility and $\mathbf{b}$ is a Nash equilibrium with a total payment of $D \cdot \frac{1}{MN+1} = \frac{2MN+1}{MN+1}$.

On the other hand, assume that there is no such a partition of S such that the sum of the numbers in each subset is N. Consider any Nash equilibrium $\mathbf{b} = \{b_0, b_1, \ldots, b_n, b_{n+1}\}$. If L_{n+1} is a winner, then the price to each winner is at least 1 and we are done. Thus, it is safe to assume that L_{n+1} is not a winner. It follows that L_0 must be a winner. Let L_j be the last winner in $\mathbf{b}$. If L_j exhausts her budget, as $D - a_0 = MN$, the set of winners excluding L_0 defines a partition of S with sum N, a contradiction to our assumption. Hence, it suffices to consider the case where L_j does not exhaust her budget. By the rule of `PROSPER`, the price to each winner is b_j. It can be seen that $b_j > 0$ (otherwise, as argued above, L_0 can increase her bid to 1 to obtain a positive utility). In addition, if there is L_i, $x_i \in S$, such that L_i is not a winner, then L_i can reduce her bid to $b_j - \epsilon$ to be a winner with positive payment and utility, a contradiction. Thus, all lenders $L_1, \ldots, L_n$ are winners. As $x_i \le N$ for $i = 1, \ldots, n$, the last winner L_j has to be L_0. In this case, by the property of Nash equilibrium, $b_j \ge 1$, which implies that the total payment to winners is at least $1 \cdot D = 2MN + 1$.

Hence, it is NP-hard to distinguish whether the total payment of the cheapest Nash equilibrium is smaller than or equal to $\frac{2MN+1}{MN+1}$ or at least $2MN+1$. As $\frac{2MN+1}{(2MN+1)/(MN+1)} = MN + 1$, it is NP-hard to approximate the total payment of the cheapest Nash equilibrium within a ratio of $\Omega(M) = \Omega(f(n))$. $\square$

The computation of a worst Nash equilibrium is NP-hard as well, as the following result shows.

THEOREM 3.2. *The computation of a worst Nash equilibrium of* `PROSPER` *is NP-hard. Furthermore, it does not admit any approximation algorithm within any ratio, unless P=NP.*

The reductions used to prove Theorems 3.1 and 3.2 above illustrate an interesting fact about `PROSPER`: While no lender with $r_i > p$ can be a winner in a Nash equilibrium with price p, the converse is not true: not every lender with $r_i < p$ need be a winner. This is in contrast with other uniform price mechanisms, as we will see in §5.

3.1 Characterizing Equilibria

The hardness in the previous results arises entirely because losers can bid strictly higher than their true rate. We now consider equilibria where *losers bid exactly their true interest rate*[6] — for this case, we characterize exactly the set of prices that can arise in a Nash equilibrium. Note that *winners* can, and indeed do, bid less than their true value in certain equilibria (see Examples 3.1 and 3.2)

DEFINITION 3.1 (INDEX α, $\alpha + 1$ AND β). *Given bid profile $\mathbf{r}$, we use α to denote the index of the last VCG winner and $\alpha + 1$ to denote the index of the first VCG loser.*

For each $L_j \in \Delta$, let L_{β_j} be the last VCG winner when the set of lenders is restricted to $\{L_1, \ldots, L_{j-1}, L_{j+1}, \ldots, L_n\}$, i.e., it is the smallest index k such that $\sum_{i=1, i \ne j}^{k} a_i \ge D$. Define $\beta = \max_{L_j \in \Delta} \beta_j$.

Here β is the index of the last lender whose bid affects the VCG payment of some VCG winner (alternately, it is the largest index of lenders who enter the VCG solution if any VCG winner is removed from the market). For example, suppose that $D = 11$, with 5 lenders $L_1, \ldots, L_5$, with respective budgets $a_1 = 6$, $a_2 = 5$, $a_3 = 4$, $a_4 = 3$, $a_5 = 12$, and interest rates $r_1 < \cdots < r_5$. Then $\Delta = \{L_1, L_2\}$, $\alpha = 2$, $\alpha + 1 = 3$ and $\beta = 4$. Now if instead $a_3 = 2$, then $\beta = 5$. Note that α, $\alpha + 1$ and β are independent to the real bids b_i and only depend on the true interests r_i of lenders. An important implication is that $L_\beta \notin \Delta$. As we will see, α, $\alpha + 1$ and β play a crucial role in characterizing the equilibria of `PROSPER`.

We now prove a sequence of lemmas about the equilibria in `PROSPER`.

LEMMA 3.1. *In any Nash equilibrium $\mathbf{b}$ with price p, any lender L_i with $r_i < p$ is a winner.*

LEMMA 3.2. *The price p in any Nash equilibrium $\mathbf{b}$ of* `PROSPER` *satisfies $r_{\alpha+1} \le p \le r_\beta$. Furthermore, $p = r_j$ for some L_j with $r_{\alpha+1} \le r_j \le r_\beta$.*

[6]As before one can argue that losers should indeed bid their true interest — while bidding strictly higher than r_i never causes negative utility, it can potentially hurt the lender's chance of obtaining positive utility if some winner exits the market.

The lemma above is crucial and characterizes the set of possible prices that can arise in an equilibrium. The next lemma follows easily from the previous ones.

LEMMA 3.3. *For any Nash equilibrium with price p, there exists a Nash equilibrium with the same price and where all lenders in Δ are winners.*

The lemmas above clearly bring out the difference between equilibria where $b_i = r_i$ for losers, and those where losers bid $b_i \geq r_i$ — as the example in the hardness reduction of Theorem 3.1 shows, these claims are false for the latter, more general case.

It is tempting to think that we may assume without loss of generality that every winner other than the last winner bids her true rate, since the actual bid value of such a winner affects neither the set of winners nor the price. However, as the example below shows, this is not true: while increasing the bid to the true rate indeed does not change the allocation or the price, the resulting profile is no longer an equilibrium.

EXAMPLE 3.1. *Suppose $D = 11$. There are four lenders with $a_1 = 5, r_1 = 0.5$; $a_2 = 10, r_2 = 1$; $a_3 = 5, r_3 = 2$ and $a_4 = 10, r_4 = 7.1$. For example, bid profile $\mathbf{b} = (2, 0, 2, 7.1)$ is a Nash equilibrium with allocation $x_1 = 1$ and $x_2 = 10$. Actually, it can be seen that it is a cheapest Nash equilibrium as well. Note that if L_2 was to bid at least her true value 1, L_1 has no incentive to increase her bid any more, and thus L_2 will have to increase her bid to 7.1 to maximize her utility.*

Note that in the above example, there is no equilibrium at price 2 with every winner bidding at least her true rate. While winners' bids in a Nash equilibrium can, in general, be quite complicated, the following simple equivalence holds.

LEMMA 3.4. *Suppose that $\mathbf{b} = (b_1, \ldots, b_k, r_{k+1}, \ldots, r_n)$ is a Nash equilibrium where $L_1, \ldots, L_k$ are winners and L_k is the last winner. Then the profile $\mathbf{b}' = (0, \ldots, 0, b_k, r_{k+1}, \ldots, r_n)$ (i.e., every winner except the last winner bids 0) constitutes a Nash equilibrium with the same allocation and price.*

PROOF. It is easy to see that both $\mathbf{b}$ and $\mathbf{b}'$ have the same allocation and price to winners, which is at least b_k. By Lemma 3.1, we know that for any L_i, $i = k+1, \ldots, n$, $r_i \geq b_k$. As all losers and the last winner L_k bid the same value in $\mathbf{b}$ and $\mathbf{b}'$, it is easy to see that no lender has an incentive to increase her bid in $\mathbf{b}'$ to obtain more utility. In addition, by the fact that $\mathbf{b}$ is a Nash equilibrium, it can be seen that lenders $L_1, \ldots, L_{k-1}, L_{k+1}, \ldots, L_n$ cannot obtain more utility by decreasing their bid. For L_k, even if L_k is able to reduce her bid down to 0, either the price drops to 0 as well (if L_k does not exhaust her budget in $\mathbf{b}$) or her allocation does not change (if L_k exhausts her budget in $\mathbf{b}$). Thus, L_k cannot obtain more utility by decreasing her bid. Therefore, $\mathbf{b}'$ is a Nash equilibrium with the same allocation and price as $\mathbf{b}$. $\square$

The lemmas above give us the following algorithm to compute Nash equilibria. The algorithm essentially checks all possible *pairs of last winners and prices*, for every lender in Δ and every price r_j, $r_{\alpha+1} \leq r_j \leq r_\beta$ — note that the price in each $\mathbf{b}(k, r_j) \in S$ is exactly r_j.

ALG-PROSPER

1. Let $S = \emptyset$.
2. For each $L_k \in \Delta$ and $r_j \in \{r_\ell \mid L_\ell : \ r_{\alpha+1} \leq r_\ell \leq r_\beta\}$
   - define a bid profile $\mathbf{b}(k, r_j)$ where
     - $b_i = 0$ for each L_i with $r_i < r_j$, $i \neq k$;
     - $b_k = r_j$;
     - $b_i = r_i$ for each L_i with $r_i \geq r_j$, $i \neq k$;
   - If L_k is the last winner in $\mathbf{b}(k, r_j)$ and it is a Nash equilibrium, let $S \leftarrow S \cup \mathbf{b}(k, r_j)$.
3. Output S.

As there are n lenders in the market, $\beta - \alpha < n$, ALG-PROSPER runs in polynomial time.

THEOREM 3.3. *Set S computed by* ALG-PROSPER *constitutes Nash equilibria with all possible prices.*

PROOF. Consider any Nash equilibrium $\mathbf{b} = (b_1, \ldots, b_k, r_{k+1}, \ldots, r_n)$ where $L_1, \ldots, L_k$ are winners and L_k is the last winner. Let p be the price of $\mathbf{b}$ and assume without loss of generality that $b_k \leq r_{k+1} \leq \cdots \leq r_n$. Note that $r_{\alpha+1} \leq p \leq r_\beta$. By Lemma 3.4, it suffices to consider Nash equilibrium $\mathbf{b}' = (0, \ldots, 0, b_k, r_{k+1}, \ldots, r_n)$. Note that the lender L_k belongs to Δ: by Lemma 3.3, all lenders of Δ are winners. Since those lenders are enough to fulfill the demand, the last winner must be one of them. As $\mathbf{b}$ is a Nash equilibrium, $p = r_{k+1}$ anyway (no matter whether L_k exhausts her budget or not). Hence, $\mathbf{b}(k, r_{k+1})$ is a Nash equilibrium as well, *i.e.* $\mathbf{b}(k, r_{k+1}) \in S$. $\square$

Why do we need to check all possible pairs of last winners and prices? The example below shows that this is necessary in general.

EXAMPLE 3.2. *Let $D = 12$. There are five lenders with $r_1 = 0, a_1 = 10$; $r_2 = 1, a_2 = 3$; $r_3 = 2, a_3 = 1$; $r_4 = 2.9, a_4 = 1$ and $r_5 = 4, a_5 = 12$, respectively. In this example, it can be seen that the cheapest Nash equilibrium is $\mathbf{b} = (0, 2.9, 0, 2.9, 5)$ with a total payment of $12 \cdot 2.9 = 34.8$, where L_1, L_2, L_3 are winners with allocation $10, 1, 1$, respectively. An interesting fact of $\mathbf{b}$ is that L_3 is even not a VCG winner. To obtain positive utility, L_3 reduces her bid to 0, which drives L_2 to increase her bid to 2.9. However, if winners have to bid at least their true value, L_2 only wants to increase her bid to $r_3 = 2$ with an allocation of 2. In this case, the utility of L_1 is $10 \cdot 2 = 20$ and she will increase her bid to $r_5 = 4$ to obtain an utility of $(D - a_2 - a_3 - a_4) \cdot 4 = 7 \cdot 4 = 28$.*

Essentially, there can exist *bullying equilibria*, where lenders with high interest rates (L_3 in the above example) bid zero and force lender L_2 to *hold up* an interest rate that would not otherwise belong to the set of utility maximizing bids for her. Restricting winners to bid at least their true value removes such equilibria and allows us to provide a sharper characterization, as we will see in the following subsection.

The characterization in Theorem 3.3 easily gives the cheapest and worst Nash equilibria as the smallest and highest prices in S. In fact, the worst Nash equilibrium always has price r_β.

THEOREM 3.4. *Let $L_j \in \Delta$ be the lender where L_β is a VCG winner by lenders $\{L_1, \ldots, L_{j-1}, L_{j+1}, \ldots, L_n\}$. Then the bid profile $\mathbf{b} = (b_1, \ldots, b_n)$, where $b_i = 0$ if $r_i < r_\beta$ and $i \neq j$, $b_j = r_\beta$ and $b_i = r_i$ if $r_i \geq r_\beta$ and $i \neq j$, is a worst Nash equilibrium with price r_β.*

3.2 Winners Bid at Least Their Private Rates

We have, until now, placed no restrictions on the winners' bids at all — specifically, a winner may bid strictly less than her true interest rate to increase her allocation and utility in an equilibrium. (For instance, in Example 3.1, by bidding 0, L_2 obtains an utility of $10 \cdot (2-1) = 10$.) However, such a bidding strategy also carries the risk of negative utility: suppose a new lender L_5 with budget $a_5 = 5$ and bid $b_5 = 0.5$ enters the market in Example 3.1. Then L_2 remains a winner but receives price 0.5, which is less than her true interest rate $r_2 = 1$, leading to negative utility. This motivates the study of the special case where winners bid no less than their true interest rates (we continue to assume that losers bid truthfully).

This allows us to consider a much smaller subset of pairs of last winners and prices to characterize all equilibria, as outlined below. Starting with the profile $\mathbf{r}$ of true interest rates, let V_i be the subset of values to which each lender $L_i \in \Delta$ is willing to increase her bid to be the last winner, given the bids of other lenders. We will show that if any of these lenders deviates to bid one of these values, the set of values to which other lenders want to move up does not change much at all — it either shrinks or stays the same, but no new value is added to it. This will allow us to show that the set of prices in equilibria is a subset of $\bigcup_{L_i \in \Delta} V_i$.

Our first lemma is very similar to Lemma 3.4.

LEMMA 3.5. *Suppose that $\mathbf{b} = (b_1, \ldots, b_k, r_{k+1}, \ldots, r_n)$ is a Nash equilibrium where $L_1, \ldots, L_k$ are winners and L_k is the last winner.*

- *If L_k does not exhausts her budget, then the profile $\mathbf{b}' = (r_1, \ldots, r_{k-1}, b_k, r_{k+1}, \ldots, r_n)$ (i.e., every lender except L_k bids her true interest) is a Nash equilibrium with the same allocation and price.*
- *If L_k exhausts her budget, then $\mathbf{b}' = (r_1, \ldots, r_k, r_{k+1}, \ldots, r_n)$ (i.e., every lender bids her true interest) is a Nash equilibrium with the same allocation and price.*

Consider a bid profile $\mathbf{b} = (b_1, \ldots, b_n)$. For each lender L_i, define a set of values $V_i(\mathbf{b})$ where $b'_i \in V_i(\mathbf{b})$ if (i) $b'_i > b_i$, (ii) when bidding b'_i, L_i is the last winner and does not exhaust her budget, and (iii) b'_i is the bid that maximizes her utility (*i.e.* $u_i(b'_i) \geq u_i(b)$ for any b, and in particular, $u_i(b'_i) \geq u_i(b_i)$). If $V_i(\mathbf{b}) \neq \emptyset$, we say that L_i is *weakly* willing to increase her bid. Intuitively, if L_i is weakly willing to increase her bid, then either she can obtain more utility by increasing her bid or the utility of other lenders can be increased without hurting her own utility.

LEMMA 3.6. *Given a bid profile $\mathbf{r} = \{r_1, \ldots, r_n\}$, let $T = \{L_i \mid V_i(\mathbf{r}) \neq \emptyset\}$. For each $L_j \in T$, let $b_j \in V_i(\mathbf{r})$ be an arbitrary value in $V_i(\mathbf{r})$. Define a profile*

$$\mathbf{b}(j, b_j) = (r_1, \ldots, r_{j-1}, b_j, r_{j+1}, \ldots, r_n)$$

and let

$$T(j, b_j) = \{L_i \mid V_i(\mathbf{b}(j, b_j)) \neq \emptyset\}$$

be the set of lenders who are weakly willing to increase their bids in $\mathbf{b}(j, b_j)$. Then for any $L_j \in T$,

(a) *$T(j, b_j) \subseteq T$.*

(b) *$L_i \notin T(j, b_j)$ for any $L_i \in T$, $i \neq j$, with $b_i \leq b_j$.*

(c) *$V_i(\mathbf{b}(j, b_j)) = V_i(\mathbf{r})$ for any $L_i \in T(j, b_j)$, $i \neq j$.*

In the above lemma, Fact **(a)** tells us that the set of lenders who are weakly willing to increase their bid does not expand if some winner increases her bid. Fact **(b)** says if a lender $L_j \in T$ with larger b_j increases her bid, then all other lenders $L_i \in T$ with smaller b_i will not increase their bids any more. Fact **(c)** says that if L_i is still weakly willing to increase her bid after another lender $L_j \in T$ increases her bid, the set of values to which L_i is weakly willing to move will not change.

The following corollary follows immediately.

COROLLARY 3.1. *Any sequence of moves starting with $\mathbf{r} = \{r_1, \ldots, r_n\}$, where a move consists of a lender $L_i \in T(\mathbf{b})$ increasing her bid to any $b \in V_i(\mathbf{b})$ where $\mathbf{b}$ is the current bid profile, converges to a Nash equilibrium.*

Let $V = \{v_1, v_2, \ldots, v_m\} = \bigcup_{L_i \in T} V_i(\mathbf{r})$ where values are ordered by $v_1 < v_2 < \cdots < v_m$. Note that $m \leq n$. For each $v_j \in V$, let $f(j)$ be the index of the lender where $v_j \in V_{f(j)}(\mathbf{r})$. (If there are multiple such lenders, pick one arbitrarily.) That is, v_j is a value to which lender $L_{f(j)}$ is weakly willing to move. Given Lemma 3.6, we can characterize the cheapest and worst Nash equilibrium easily.

THEOREM 3.5. *The following results hold:*

- *The cheapest Nash equilibrium is either $\mathbf{r}$ or given by the smallest index ℓ, $1 \leq \ell \leq m$, where $\mathbf{b}(f(\ell), v_\ell)$ is a Nash equilibrium.*
- *The worst Nash equilibrium is either $\mathbf{b}(f(m), v_m)$ or $\mathbf{r}$ (if $V = \emptyset$).*

4. COMPARING PROSPER AND VCG

In this section, we compare the total payment of VCG with that of `PROSPER`, in a setting where losers bid at least their true interest rate and with no restrictions on winners' bids (*i.e.*, the most general setting considered in § 3). For any given instance, let `CNE(PROSPER)` and `WNE(PROSPER)` denote the total payment of the cheapest and worst Nash equilibrium of `PROSPER`, respectively. Let `VCG` denote the total payment of VCG.

THEOREM 4.1. *The following inequalities hold*

$$\frac{1}{D} \cdot \texttt{CNE(PROSPER)} \leq \texttt{VCG} \leq O(\log D) \cdot \texttt{CNE(PROSPER)}$$

PROOF. Consider any cheapest Nash equilibrium $\mathbf{b}^* = (b_1^*, \ldots, b_k^*)$ of the `PROSPER`. Let x^* be the vector of allocations and p^* be the price to all winners in $\mathbf{b}^*$. Assume without loss of generality that lenders are ordered by $r_1 \leq \cdots \leq r_n$.

By Lemma 3.2, we know $\texttt{CNE(PROSPER)} \leq D \cdot r_\beta$. On the other hand, by the definition of β, r_β will be counted in the total payment of some VCG winner in Δ (recall that Δ is the set of winners by VCG), which implies that $r_\beta \leq \texttt{VCG}$.

Hence, $\texttt{CNE}(\texttt{PROSPER}) \le D \cdot r_\beta \le D \cdot \texttt{VCG}$. It remains to prove the second inequality of the claim.

Let x_i be the allocation of each L_i by VCG. As D is the total demand, we know $x_i \le D$. For each $L_i \in \Delta \setminus \{L_\alpha\}$, define an ordered multi-set S_i with $|S_i| = x_i$ by

$$S_i = \Big\{ \overbrace{r_\alpha, \ldots, r_\alpha}^{\text{total of } a_\alpha - x_\alpha}, \overbrace{r_{\alpha+1}, \ldots, r_{\alpha+1}}^{\text{total of } a_{\alpha+1}}, \overbrace{r_{\alpha+2}, \ldots, r_{\alpha+2}}^{\text{total of } a_{\alpha+2}}, \ldots\ldots \Big\}$$

That is, S_i contains $a_\alpha - x_\alpha$ many r_α's, $a_{\alpha+1}$ many $r_{\alpha+1}$'s, $a_{\alpha+2}$ many $r_{\alpha+2}$'s, and so on, until the size of S_i is x_i. Define an ordered multi-set S_α with $|S_\alpha| = x_\alpha$ by

$$S_\alpha = \Big\{ \overbrace{r_{\alpha+1}, \ldots, r_{\alpha+1}}^{\text{total of } a_{\alpha+1}}, \overbrace{r_{\alpha+2}, \ldots, r_{\alpha+2}}^{\text{total of } a_{\alpha+2}}, \ldots\ldots \Big\}$$

That is, S_α contains $a_{\alpha+1}$ many $r_{\alpha+1}$'s, $a_{\alpha+2}$ many $r_{\alpha+2}$'s, and so on, until the size of S_α is x_α. By the rule of VCG, we know that the payment to each winner $L_i \in \Delta$ is the sum of elements in S_i.

For each S_i, denote its σ-th element by $f_i(\sigma)$, $\sigma = 1, \ldots, x_i$, and let $\phi_i = \arg\max_{\sigma=1,\ldots,x_i} (x_i - \sigma + 1) f_i(\sigma)$. That is, ϕ_i is the index that gives the maximum payment (not utility!) obtainable if L_i increases her interest unilaterally. Let $\lambda_i = (x_i - \phi_i + 1) f_i(\phi_i)$. Hence, the total payment of VCG satisfies

$$\begin{aligned}
\texttt{VCG} &= \sum_{L_i \in \Delta} \sum_{\sigma=1}^{x_i} f_i(\sigma) \\
&= \sum_{L_i \in \Delta} \left(1 \cdot f_i(x_i) + \frac{1}{2} \cdot 2 f_i(x_i - 1) + \cdots + \frac{1}{x_i} \cdot x_i f_i(1) \right) \\
&\le \sum_{L_i \in \Delta} \left(\left(1 + \frac{1}{2} + \cdots + \frac{1}{x_i} \right) \cdot \max_{\sigma=1,\ldots,x_i} (x_i - \sigma + 1) f_i(\sigma) \right) \\
&= \sum_{L_i \in \Delta} O(\log x_i) \cdot \lambda_i \\
&\le O(\log D) \sum_{L_i \in \Delta} \lambda_i
\end{aligned}$$

On the other hand, consider the price p^* to all winners in $\mathbf{b}^*$. We divide the lenders in Δ into two groups:

$$\Delta_1 = \{L_i \in \Delta \mid p^* < f_i(\phi_i)\}$$

and

$$\Delta_2 = \{L_i \in \Delta \mid p^* \ge f_i(\phi_i)\}$$

For each $L_i \in \Delta_1$, we claim that the total payment (not utility!) that L_i obtains in $\mathbf{b}^*$ is at least λ_i. Otherwise, Assume that there is $L_j \in \Delta_1$ such that the total payment that L_j obtains in $\mathbf{b}^*$ is smaller than λ_j. Note that the utility of L_j in $\mathbf{b}^*$ is $u_j(\mathbf{b}^*) = x_j^* \cdot (p^* - r_j) = x_j^* p^* - x_j^* r_j$, where $x_j^* p^*$ is the total payment that L_j obtains. By the assumption, we know $x_j^* p^* < \lambda_j$. As $b_i^* \ge r_i$ for any loser L_i, when L_j increases her bid to $f_j(\phi_j)$ in $\mathbf{b}^*$, the total payment she obtains is at least λ_j. On the other hand, when the bid of L_j increases, the allocation of L_j, x_j^*, is not increasing. Since

$$\text{utility} = \text{total payment} - \text{allocation} \times \text{true interest}$$

we know that the utility of L_j increases, a contradiction to the fact that $\mathbf{b}^*$ is a Nash equilibrium. It follows that

$$\texttt{CNE}(\texttt{PROSPER}) \ge \sum_{L_i \in \Delta_1} \text{total payment to } L_i \ge \sum_{L_i \in \Delta_1} \lambda_i$$

For lenders in Δ_2, observe that

$$\begin{aligned}
\texttt{CNE}(\texttt{PROSPER}) &= D \cdot p^* \\
&\ge p^* \cdot \sum_{L_i \in \Delta_2} x_i \\
&\ge p^* \cdot \sum_{L_i \in \Delta_2} (x_i - \phi_i + 1) \\
&\ge \sum_{L_i \in \Delta_2} (x_i - \phi_i + 1) f_i(\phi_i) \\
&= \sum_{L_i \in \Delta_2} \lambda_i
\end{aligned}$$

Therefore,

$$\texttt{CNE}(\texttt{PROSPER}) \ge \frac{1}{2} \sum_{L_i \in \Delta} \lambda_i \ge \frac{1}{O(\log D)} \texttt{VCG}$$

which completes the proof of the theorem. □

The inequalities in the above theorem are tight. Consider the following two examples:

- Let $D = 10m + 1$, where m is an arbitrary positive integer. There are six lenders with budget $a_0 = 5m+1$, $a_1 = m$, $a_2 = a_3 = a_4 = 3m$, $a_5 = 11m$ and interest $r_0 = r_1 = r_2 = r_3 = r_4 = 0$ and $r_5 = 1$. It is easy to see that $\texttt{VCG} = 1$, where the only lender that obtains positive utility is L_0 who gets a payment of $r_5 = 1$. Note that $a_1 + a_2 + a_3 + a_4 = 10m$ and there is no way to partition $\{L_1, L_2, L_3, L_4\}$ into two groups such that the sum of budgets of each partition is $5m$. By the reduction shown in Theorem 3.1, the cheapest Nash equilibrium has a price 1 to each winner, which implies that the total payment is D. Hence, $\texttt{CNE}(\texttt{PROSPER}) = D \cdot \texttt{VCG}$.

- Let $D = n$. There are $n+1$ lenders $L_0, L_1, \ldots, L_n$ with budget $a_0 = n$ and $a_i = 1$ for $i = 1, \ldots, n$. Let $r_0 = 0$ and $r_i = \frac{n}{n-i+1}$ for $i = 1, \ldots, n$. In VCG, L_0 wins and the total payment is $\sum_{i=1}^n r_i a_i = \sum_{i=1}^n \frac{n}{n-i+1} = O(n \log n)$. It can be seen that the profile $(r_0, r_1, \ldots, r_n)$ is a Nash equilibrium. Therefore, the total payment is $D \cdot r_1 = n$ and $\texttt{VCG} = O(\log D) \cdot \texttt{CNE}(\texttt{PROSPER})$.

For this general setting, *i.e.*, where losers can bid higher than their true value, the worst-case ratio between the worst Nash equilibrium of PROSPER and VCG can be arbitrarily large as the reduction in the proof of Theorem 3.2 shows. If losers bid their interest rate truthfully, we have the following result, similar to Theorem 4.1 (again, the bounds are tight).

THEOREM 4.2.

$$\frac{1}{D} \cdot \texttt{WNE}(\texttt{PROSPER}) \le \texttt{VCG} \le O(\log D) \cdot \texttt{WNE}(\texttt{PROSPER})$$

5. OTHER UNIFORM PRICE MECHANISMS

PROSPER is a uniform price mechanism, meaning that all winning lenders receive the same price. How does it compare to other uniform price mechanisms? Here there are two natural candidates — pay all winners the *bid of the last winner* (denoted by BLW), and pay all winners the *bid of* *the first loser* (denoted by BFL). Both mechanisms have the same allocation rule as PROSPER (as in Definition 2.1), but a slightly different pricing rule. The mechanism BLW offers a price equal to the bid of the last winner, while BFL offers a price equal to the bid of the first loser. Note that the price of PROSPER is either that of BLW or BFL, depending on whether or not the last winner exhausts her budget.

If all "items" were identical, meaning that every lender had a budget of one, BFL would be identical to VCG, which produces an *efficient allocation* amongst lenders. As the following example shows, however, BFL is in fact very different from VCG in terms of efficient allocation: since the price is determined by the bid of the first loser, every winner has an incentive to bid as low as possible to increase her allocation when the total budget of winners is greater than the demand. Specifically, as long as the bid of the first loser is at least her true interest rate, a winner loses nothing by bidding as low as 0 to improve her allocation.

EXAMPLE 5.1. *Let $D = 11$. There are four lenders with budgets $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_4 = 10$ and interest rates $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, $r_4 = 4$, respectively. The* only *equilibrium of* BFL *has L_1, L_2, L_3 bidding 0 and the losing lender L_4 bidding her true value, and the allocation determined according to the given tie-breaking rule. (As otherwise, the lender not exhausting her budget can always reduce her bid to 0 and increase her allocation,* with no change in price *which remains at $b_4 = 4$.)*

Thus, which winner has leftover budget is entirely determined by the tie-breaking rule, which can be, in general, arbitrary: even when the winners of BFL are exactly the VCG winners, lenders with lower interest rates do not necessarily receive a better allocation, leading to inefficiency. Note that this does not happen in either PROSPER or BLW, where the last winner never bids below her true interest rate[7]. Specifically, in the above example, both PROSPER and BLW have efficient equilibria, while BFL does not.

We now investigate BLW. As the following results show, BLW is very similar to PROSPER: the set of equilibria of BLW is a subset of that of PROSPER, and when losers are restricted to bid their true values (as in § 3.1), the equilibria are identical.

Similar to Lemma 3.4 (and its proof), we have the following result.

LEMMA 5.1. *Suppose $\mathbf{b} = (b_1, \ldots, b_k, b_{k+1}, \ldots, b_n)$ is a Nash equilibrium of* BLW, *where $L_1, \ldots, L_k$ are winners and L_k is the last winner. Then the profile $\mathbf{b}' = (0, \ldots, 0, b_k, r_{k+1}, \ldots, r_n)$ (i.e., every winner bids 0 except L_k) constitutes a Nash equilibrium with the same allocation and price.*

THEOREM 5.1. *Any bid profile that is an equilibrium of* BLW *is also an equilibrium in* PROSPER*. Moreover, if we restrict ourselves to bid profiles where losers bid their true value, every equilibrium in* PROSPER *is also one in* BLW*, so that both sets of equilibria are identical.*

When losers can bid any value greater equal their true interest rates (that is, they are not restricted to bidding truthfully), the cheapest Nash equilibrium of PROSPER can be a factor D smaller than the cheapest equilibrium of BLW; conversely, the worst Nash equilibrium of PROSPER can be arbitrarily larger than the worst equilibrium of BLW. This follows directly from the examples of the hardness reductions in Theorems 3.1 and 3.2.

[7]Strictly speaking, this holds for PROSPER only when the sum of the budgets of winners exceeds demand D. If it is equal to D, the last winner exhausts her budget anyway, and bidding lower than her true interest rate does not change anything.

6. DYNAMIC PROSPER MECHANISM

Lenders can actually *change* their bids in Prosper after observing the current allocation and price, until the auction is closed. The actual process used by Prosper to clear each loan is as follows: bidding starts at the reserve rate specified by the borrower. Lenders submit their budgets and bids, and an allocation and a price is computed according to the one-shot auction PROSPER (defined in § 2). Lenders may submit new bids until the auction closes, as long as the new bid is at least a minimum increment (0.05 points in Prosper) below the current price. The allocation and price are recomputed (according to PROSPER) each time when a lender changes her bid. The winners' allocation and price are announced publicly at each time, as also the bids of losers (the budgets of all lenders are displayed publicly throughout the auction). Note that the real bids of the winners are not revealed publicly.

Our static auction model described in § 2 is easily extended to account for this dynamism. For simplicity, we will consider discrete times $t = 0, 1, \ldots, T$. At $t = 0$, the borrower publicly announces the demand D and the reserve interest rate R. Each lender L_i submits an initial bid at $t = 1$, $b_i^{(1)}$; at subsequent times, lenders may decide to either lower their bid, or to maintain their most recent offer. We only allow bids that are multiples of the minimum increment $\epsilon > 0$ (equivalently, if a lender wants to reduce her bid, she must reduce it by at least ϵ). The budget a_i of each lender L_i is common knowledge. At every time t, the one-shot auction PROSPER is used to determine the allocation and price for the bid profile $(b_1^{(t)}, \ldots, b_n^{(t)})$ (as before, we assume a fixed, pre-announced tie-breaking rule). We assume without loss of generality that the bid profile must not be equal in two consecutive rounds, *i.e.* $(b_1^{(t)}, \ldots, b_n^{(t)}) \neq (b_1^{(t+1)}, \ldots, b_n^{(t+1)})$ [8]. The winners at time t are announced publicly, as well as the price offered to winners and the bids of losers. The final outcome of the auction is the outcome of the last one-shot auction PROSPER at time T, and the price at that time is called *final price*. We refer to this dynamic process as PROSPER DYNAMICS. In this section, we index the lenders so that $r_1 \leq r_2 \leq \cdots \leq r_n$.

We first provide bounds on the final price under a very general and plausible assumption on bidding behavior: Consider a lender L_i whose utility is zero in the current round — if decreasing her bid will *strictly* increase her utility, assuming other lenders' bids remain unchanged, she will do so. This is a natural assumption, since if the other bids remain unchanged or decrease, L_i cannot get a positive utility anyway. Naturally lenders do not wish to end up with a negative utility, so similar to § 3 we also assume that they never bid less than their true interest. Note that VCG winners are al-

[8]Since bids are decreasing and bounded below by zero, they must converge in a finite number of rounds for any given $\epsilon > 0$, no matter what strategy lenders decide to follow; we assume that T is large enough so as to allow convergence in all cases.

ways among the winning lenders of PROSPER DYNAMICS, however there may be more winners. We bound the final price as stated below.

THEOREM 6.1. *The final price of* PROSPER DYNAMICS *is between r_α and r_β.*

Note that the lower bound is slightly different from Lemma 3.2 for the equilibria of PROSPER, where the price is bounded below by $r_{\alpha+1}$.

We now consider lenders with two special types of bidding behaviors: *myopic greedy* behavior and *conservative* behavior. We will show that, for both cases, the final price is more constrained, to be either the lowest or highest possible value in the range of possible prices.

Myopic greedy lenders try to maximize their utility in the next round, under the assumption that the price of the current round remains unchanged. Such lenders choose to either keep the same bid for the next round when they cannot increase their utility, or decrease their bid just below the current price when this would allow them to get more allocation. Formally, a sequence of bids corresponds to a *myopic greedy* behavior when, for all L_i, $b_i^{(1)} = R$; and for all $t > 1$, if $x_i^{(t)} < a_i^{(t)}$ and $p^{(t)} - \epsilon \geq r_i$, then $b_i^{(t+1)} = p^{(t)} - \epsilon$; otherwise $b_i^{(t+1)} = b_i^{(t)}$. We show that, when all lenders follow myopic greedy strategies, the price converges to the lowest possible rate.

THEOREM 6.2. *If all lenders are myopic greedy, then the winners are Δ (i.e. the set of VCG winners) and the final price is $r_{\alpha+1}$ when all winners exhaust their budget and r_α otherwise.*

Interestingly, when all lenders are myopic greedy, the final outcome is exactly the same as that in PROSPER when all lenders bid their true interest rate (of course, this need not be an equilibrium bid profile in PROSPER).

Conservative lenders attempt to maximize their final utility at the last round *under worst-case assumptions* about other lenders' true interest rates. The worst-case scenario for a lender L_i occurs when every other winner has a true interest less than L_i's interest rate. When this is the case, a lender should never decrease her bid when she has a positive utility in the current round, otherwise she can decrease it by the minimum increment ϵ (as long as it is above her true interest rate). Formally, a sequence of bids corresponds to a *conservative* bidding behavior when, for all L_i, $b_i^{(1)} = R$, and for all $t > 1$, $b_i^{(t+1)} = b_i^{(t)}$ if $x_i^{(t)} > 0$ or $b_i^{(t)} = r_i$; and $b_i^{(t+1)} = b_i^{(t)} - \epsilon$ otherwise. When all lenders follow conservative bidding strategies, the final price is the maximum possible price.

THEOREM 6.3. *If all lenders are conservative, then the final price is no less than $r_\beta - \epsilon$.*

A natural question to ask is how the borrower's payment in PROSPER DYNAMICS compares with that in VCG (the VCG mechanism also has a dynamic implementation for this setting, see [5]). However, we show that no mechanism dominates the other, in the sense that the total payment to the lenders in PROSPER DYNAMICS can be larger than that in VCG mechanism and vice-versa, depending on bidder behavior.

- Let $D = 15$ and $R = 10$. There are three lenders L_1, L_2, L_3 with respective budgets $a_1 = 14, a_2 = 2, a_3 = 20$ and interest rates $r_1 = \delta, r_2 = 2\delta, r_3 = 1$. The VCG allocation is then $x_1 = 14, x_2 = 1, x_3 = 0$, and the VCG payment to L_1 is $2\delta \cdot 2 + 13 \cdot 1$, while the payment to L_2 is 1. In PROSPER DYNAMICS, if lenders play myopic greedy strategies, the price is δ and so the total payment is $15 \cdot 2\delta$, which can be arbitrary smaller than the VCG payment when δ tends to 0.
- Assume that the demand is any $D > 0$, and consider $D + 1$ lenders, with respective budgets $a_1 = D, a_2 = \cdots = a_D = 1, a_{D+1} = 2D$, and interest rates $r_1 = \delta, r_2 = 2\delta = \cdots = r_D = 2\delta, r_{D+1} = 1$. The total VCG payment is then $2\delta \cdot (D-1) + 1$, while in PROSPER DYNAMICS with conservative bidders, the total payment is $D \cdot 1$. As δ tends to 0, the ratio between PROSPER DYNAMICS and VCG payment tends to D. Note that the ratio cannot be greater than D as in all cases VCG allocates at least one unit of demand at rate r_β.

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7. REFERENCES

[1] www.lendingclub.com.

[2] www.prosper.com.

[3] www.zopa.com.

[4] A. Archer and E. Tardos. Frugal Path Mechanisms. In *Proceedings of the 13th Annual Symposium on Discrete Algorithms*, pages 991–999, 2002.

[5] L. Ausubel. An Efficient Ascending-Bid Auction for Multiple Objects. *American Economic Review*, 94(5):1452–1475, 2004.

[6] N. Chen and A. Karlin. Cheap Labor Can Be Expensive. In *Proceedings of the 18th Annual Symposium on Discrete Algorithms*, pages 707–715, 2007.

[7] E. Elkind. True Costs of Cheap Labor are Hard to Measure: Edge Deletion and VCG Payments in Graphs. In *Proceedings of the 6th ACM Conference on Electronic Commerce*, pages 108–116, 2005.

[8] E. Elkind, A. Sahai, and K. Steiglitz. Frugality in Path Auctions. In *Proceedings of the 15th Annual Symposium on Discrete Algorithms*, pages 701–709, 2004.

[9] S. Freedman and G. Jin. Do social networks solve information problems for peer-to-peer lending? evidence from prosper.com. Working Papers 08-43, NET Institute, Nov. 2008.

[10] N. Immorlica, D. Karger, E. Nikolova, and R. Sami. First-Price Path Auctions. In *Proceedings of the 6th ACM Conference on Electronic Commerce*, pages 203–212. ACM New York, NY, USA, 2005.

[11] E. Kalai and E. Lehrer. Rational Learning Leads to Nash Equilibrium. *Econometrica*, 61:1019–1019, 1993.

[12] A. Karlin, D. Kempe, and T. Tamir. Frugality of Truthful Mechanisms. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 615–626, 2005.

[13] V. Krishna. *Auction Theory.* Academic Press, 2002.
[14] K. Talwar. The Price of Truth: Frugality in Truthful Mechanisms. *Proceedings of the 20th Annual Symposium on Theoretical Aspects of Computer Science*, pages 608–619, 2003.
[15] H. Varian. Position Auctions. *International Journal of Industrial Organization*, 25(6):1163–1178, 2007.

APPENDIX

Supplementary material

A. PROOFS IN SECTION 3

A.1 Proof of Theorem 3.2

PROOF. We reduce from Partition: Given a set of integers $S = \{x_1, \dots, x_n\}$ where $\sum_{i=1}^n x_i = 2N$, can S be partitioned into two subsets such that the sum of the numbers in each subset is N? Assume without loss of generality that $1 \le x_i \le N$, for $i = 1, \dots, n$.

We construct an instance of our problem as follows: For $i = 1, \dots, n$, there is a lender L_i with budget $a_i = x_i$ and interest $r_i = 0$. Let $D = N$ and $R = 1$. We claim that it is NP-hard to distinguish whether the total payment of the worst Nash equilibrium is either 0 or N.

Assume that there is a partition of S into S_1 and S_2 such that the sum of the numbers in each subset is N. We construct a bid profile $\mathbf{b}$ as follows: Let $b_i = r_i$ if $x_i \in S_1$ and $b_i = R = 1$ if $x_i \in S_2$. Given $\mathbf{b}$, as $\sum_{i: x_i \in S_1} a_i = N = D$, the winners are those corresponding to the set S_1 and all winners exhaust their budget. Thus, the price to each winner is 1. It is easy to see that no winner is willing to increase her bid. Additionally, if any loser L_i reduces her bid to 0, even if L_i becomes a winner, since the price is reduced to 0 as well, L_i still obtains 0 utility. Hence, no lender can unilaterally increase her utility and $\mathbf{b}$ is a Nash equilibrium with a total payment of $D \cdot 1 = N$.

On the other hand, assume that there is no such a partition of S such that the sum of the numbers in each subset is N. Consider any Nash equilibrium $\mathbf{b} = \{b_1, \dots, b_n\}$. Let L_j be the last winner in $\mathbf{b}$. If L_j exhausts her budget, then the set of winners constitutes a partition of S with sum N, a contradiction to our assumption. Hence, L_j does not exhaust her budget. By the rule of `PROSPER`, the price to each winner is b_j. If $b_j > 0$, we claim that all lenders $L_1, \dots, L_n$ are winners. Otherwise, if L_i is not a winner (which implies that $b_i \ge b_j$), by reducing her bid to $b_j - \epsilon > 0$, L_i becomes a winner with price at least $b_j - \epsilon$, a contradiction to the fact that $\mathbf{b}$ is a Nash equilibrium. However, if all lenders are winners, we know that $\sum_{i: i \ne j} a_i = \sum_{i: i \ne j} x_i < D = N$. Because $\sum_{i=1}^n x_i = 2N$, we have $x_j > N$, a contradiction to our assumption. Hence, $b_j = 0$, which implies that the total payment to winners is 0.

Hence, it is NP-hard to distinguish whether the total payment of the worst Nash equilibrium is 0 or N, which implies that we do not have any approximation algorithm within any ratio, unless P=NP. □

A.2 Proof of Lemma 3.1

PROOF. The claim follows directly from the assumption that all losers bid their true interest and the definition of the mechanism: for any lender L_i with $r_i < p$, if L_i is not a winner, we know L_i bids her interest truthfully. This implies that the price p is at most r_i, a contradiction. □

A.3 Proof of Lemma 3.2

PROOF. **(a)** We start with the lower bound, $p \ge r_{\alpha+1}$. By contradiction, suppose that $p < r_{\alpha+1}$ for some Nash equilibrium. Note that since any VCG loser has an interest rate greater than p, any VCG loser is a loser. It is easy to see that $p \ge r_\alpha$ as otherwise, by the definition of α, the total demand cannot be fulfilled. If $p = r_\alpha$, the last VCG winner L_α obtains zero utility, and thus is willing to increase her bid to $r_{\alpha+1}$ to obtain positive utility. If $r_\alpha < p < r_{\alpha+1}$, by Lemma 3.1, all lenders of Δ are winners. Hence, the last winner does not exhaust her budget and would profit from increasing her bid to $r_{\alpha+1}$, a contradiction.

(b) We now deal with the upper bound, $p \le r_\beta$. By contradiction, suppose that $p > r_\beta$ for some Nash equilibrium. Let L_k be the last winner. By Lemma 3.1, we know that all lenders L_i with an interest $r_i \le r_\beta < p$ are winners. In particular, lenders in $\Delta \cup \{L_\beta\}$ are winners, implying by Definition 2.3 that $L_k \in \Delta$. This contradicts to Definition 3.1 of β.

(c) It remains to prove that $p = r_j$ for some L_j with $r_{\alpha+1} \le r_j \le r_\beta$. If $p \ne r_j$ for any L_j with $r_{\alpha+1} \le r_j \le r_\beta$, as all losers bid their true interest, the only possible case is that the last winner L_k does not exhaust her budget and hence $p = b_k$. In this case, however L_k will always increase her bid to the point which equals the bid of the first loser, a contradiction. □

A.4 Proof of Lemma 3.3

PROOF. From Lemma 3.2, $p \ge r_{\alpha+1}$. If $p > r_{\alpha+1}$, then by Lemma 3.1 all lenders with an interest rate less than or equal to $r_{\alpha+1}$ are winners, and so by definition of Δ all lenders of Δ are winners.

If $p = r_{\alpha+1}$, then by Lemma 3.1, all lenders $L_i \in \Delta$ with $r_i < r_{\alpha+1} = p$ are winners. If a VCG winner is a loser, she must bid her true interest rate by assumption, which must then be greater than or equal to p since she loses the auction, but also no greater than $r_{\alpha+1}$ since she belongs to Δ. Hence all lenders of Δ that are losers bid exactly $r_{\alpha+1}$. For any winner $L_i \notin \Delta$, $r_i \ge r_{\alpha+1}$, but since L_i gets a nonnegative utility, $r_i \le r_{\alpha+1}$, and so $r_i = r_{\alpha+1}$. In other words, any winner that does not belong to Δ gets a zero utility, and so can increase her bid to her true interest rate $r_{\alpha+1}$ without changing her utility. This gives a Nash equilibrium with the same price, and by the tie-breaking rule, the winners are exactly the lenders of Δ. □

A.5 Proof of Theorem 3.4

PROOF. We first prove that $\mathbf{b}$ is a Nash equilibrium with price r_β. Let L_k be the last winner and p be the price of $\mathbf{b}$, respectively. By the definition of β and selection of L_j, we know that $p \ge r_\beta$. If $p > r_\beta$, by the construction of $\mathbf{b}$, all lenders in $\Delta \cup \{L_\beta\}$ are winners. Since $L_\beta \notin \Delta$, we know that $L_k \in \Delta$ and L_k does not exhaust her budget, which implies that $b_k = p > r_\beta$, which is impossible. Hence $p = r_\beta$.

It is easy to see that no winner in $\mathbf{b}$ can obtain more utility by increasing her bid (this is because, for any winner L_i, if moving her bid to a value higher than $p = r_\beta$, she will not get any allocation). Additionally, all winners with either bid 0 or true interest $p = r_\beta$ and all losers do not have an incentive to decrease their bid. The only lender we need to consider is L_j. If $j \neq k$, L_j exhausts her budget in $\mathbf{b}$, and thus has no incentive to reduce her bid. If $j = k$, as $L_j = L_k$ does not exhaust her budget, then when reducing her bid, the price will be decreased as well. Hence, L_j is not willing to reduce her bid. Therefore, $\mathbf{b}$ is a Nash equilibrium.

From Lemma 3.2, we know that the price in any Nash equilibrium lies between $r_{\alpha+1}$ and r_β. Therefore this must be the worst Nash equilibrium. $\square$

A.6 Proof of Lemma 3.5

PROOF. It is easy to see that both $\mathbf{b}$ and $\mathbf{b}'$ have the same allocation and price to winners, which is at least b_k. By Lemma 3.1, we know that for any L_i, $i = k+1, \ldots, n$, $r_i \geq b_k$.

If L_k does not exhausts her budget, as all losers and the last winner L_k bid the same value in $\mathbf{b}$ and $\mathbf{b}'$, it is easy to see that no lender has an incentive to increase her bid in $\mathbf{b}'$ to obtain more utility. In addition, by the fact that $\mathbf{b}$ is a Nash equilibrium, it can be seen that lenders $L_1, \ldots, L_{k-1}, L_{k+1}, \ldots, L_n$ cannot obtain more utility by decreasing her bid. For L_k, if L_k can obtain more utility by decreasing her bid in $\mathbf{b}'$, she is willing to decrease her bid in $\mathbf{b}$ as well, a contradiction. Therefore, $\mathbf{b}'$ is a Nash equilibrium with the same allocation and price as $\mathbf{b}$.

The case where L_k exhausts her budget is similar. Hence, the claim follows. $\square$

A.7 Proof of Lemma 3.6

PROOF. For simplicity, in the proof, we denote $\mathbf{b}(j, b_j)$ by $\mathbf{b}(b_j)$ and $T(j, b_j)$ by $T(b_j)$.

(a) Consider any $L_i \in T(b_j)$. Note that the only difference between $\mathbf{r}$ and $\mathbf{b}(b_j)$ is that L_j increases her bid from r_j to b_j. Hence, the utility of all other lenders does not decrease, which implies that $u_i(\mathbf{r}) \leq u_i(\mathbf{b}(b_j))$. Furthermore, by the definition of T and $T(j)$, L_i is weakly willing to increased her bid in $\mathbf{b}(b_j)$, which implies she is weakly willing to increase her bid in $\mathbf{r}$ as well. Hence, $L_i \in T$ and $T(b_j) \subseteq T$.

(b) Assume otherwise that there is $L_i \in T$ with $b_i \leq b_j$ such that $L_i \in T(b_j)$. Further, by the definition of $V_i(\mathbf{r})$ and $\mathbf{b}(b_j)$, L_j is the last winner in $\mathbf{b}(b_j)$ and her budget is not exhausted. Hence, L_i exhausts her budget with price b_j in $\mathbf{b}(b_j)$. In $\mathbf{b}(b_i)$, however, L_i does not exhausts her budget with price b_i. As $b_j \geq b_i > r_i$, we have $u_i(\mathbf{b}(b_j)) > u_i(\mathbf{b}(b_i))$. Therefore, if $L_i \in T(b_j)$, *i.e.* L_i is weakly willing to increase her bid in $\mathbf{b}(b_j)$, the utility of L_i can be strictly increased in $\mathbf{b}(b_i)$, which contradicts the definition of b_i — the point where the utility of L_i is maximized.

(c) As $L_i \in T(b_j)$, by Fact **(a)**, we know that $L_i \in T$. If there is $b \in V_i(\mathbf{r})$ such that $b \leq b_j$, by picking $b_i = b$ for L_i, we know that $L_i \notin T(b_j)$ by Fact **(b)**, a contradiction. Hence, in $\mathbf{r}$ and $\mathbf{b}(b_j)$, L_i is weakly willing to increase her bid to the same points. (Note that when L_i bids a value higher than b_j, it does not matter if L_j bids r_j or b_j.) $\square$

A.8 Proof of Corollary 3.1

PROOF. For convergence, note that for any sequence of moves, the price keeps increasing; so in at most n steps, no lender is willing to increase her bid. At convergence, no lender wants to increase her bid. Further, as all losers bid their true interest, no lender can obtain more utility by decreasing her bid. Hence, it converges to a Nash equilibrium. $\square$

A.9 Proof of Theorem 3.5

PROOF. For the simplicity of the proof, we denote $\mathbf{b}(f(j), v_j)$ by $\mathbf{b}(j)$.

Let $\mathbf{b}^* = (b_1^*, \ldots, b_n^*)$ be a cheapest Nash equilibrium with last winner L_k. If L_k exhausts her budget in $\mathbf{b}^*$, by Lemma 3.5, we know that $\mathbf{r}$ is a cheapest Nash equilibrium as well. If L_k does not exhausts her budget, again by Lemma 3.5, it is safe to assume that all other lenders bid their true interest in $\mathbf{b}^*$. As the utility of L_k is maximized by bidding b_k^*, we know that $L_k \in T$ and $b_k^* \in V_k(\mathbf{r}) \subseteq V$ (otherwise, $\mathbf{r}$ is a Nash equilibrium). Hence, any profile $\mathbf{b}(j)$, where $v_j < b_k^*$, is not a Nash equilibrium. Consider the profile $\mathbf{b}(j)$, where $v_j = b_k^*$. By Lemma 3.6, we know that $\mathbf{b}(j)$ is a Nash equilibrium, which is the cheapest Nash equilibrium as well.

Similarly, we can prove that $\mathbf{b}(f(m), v_m)$ is a worst Nash equilibrium. $\square$

B. PROOFS IN SECTION 5

B.1 Proof of Lemma 5.1

PROOF. In any Nash equilibrium $(b_1, \ldots, b_k, b_{k+1}, \ldots, b_n)$ of `BLW`, any lender L_j with $r_j < p = b_k$ must be a winner. Indeed, if not, L_j can bid $p - \epsilon \geq r_j$ and get positive utility. So all lenders whose private interest rate is less than p must be winners, and $r_i \geq p$ for all losers L_i. By assumption, losers bid greater or equal to their true value, so $b_i \geq r_i \geq p$ as well.

As in Lemma 3.4 winners L_i, $i < k$ can replace their bid by 0 and no one has an incentive to deviate. Suppose a loser L_j decreases her bid from b_j to $r_j \leq b_j$, no winner can profit from increasing her bid because the same increase would have profited her in the previous profile as well. $\square$

B.2 Proof of Theorem 5.1

PROOF. Note that both mechanisms are identical when the last winner does not exhaust her budget. Let's consider an equilibrium $\mathbf{b} = (b_1, \ldots, b_n)$ of `BLW` where L_k is the last winner. By Lemma 5.1, we can assume without loss of generality that all lenders other than L_k bids her true interest rate.

Suppose otherwise that $\mathbf{b}$ is not an equilibrium in `PROSPER`. This means there is a profitable deviation for some lender L_i in `PROSPER`. There are the following cases.

Case 1. L_i is a winner, $i \neq k$. Her only possible profitable deviation is to increase her bid above b_k. Since $\mathbf{b}$ is an equilibrium in `BLW`, $b_k = r_{k+1}$, and therefore, by increasing her bid, at least one loser becomes a winner and L_i does not exhaust her budget anymore. In that case `PROSPER` and `BLW` compute the exact same allocation and price, which means that if L_i profits from that deviation in `PROSPER`, she would profit from the same deviation in `BLW`, which contradicts the fact that $\mathbf{b}$ is a Nash equilibrium in `BLW`.

Case 2. $L_i = L_k$. As she does not wish to deviate in `BLW`, $b_k = r_{k+1}$. In `PROSPER`, L_k cannot profit from decreasing her bid, since if she is exhausting her budget, her allocation and price remain the same. If she is not exhausting her budget, decreasing her bid can only decrease her price. In all cases, L_k cannot profit by decreasing. By an identical argument as the above case, she cannot profit by increasing her bid in `PROSPER` as well.

Case 3. L_i is a loser. We show that if there is a profitable deviation in `PROSPER`, there must be one in `BLW` as well. Increasing her bid does not increase her allocation (which is 0), and therefore does not increase her utility. Decreasing her bid leads to a price that is less than her true value, which decreases utility (since losers bid at least their true interest rate).

Therefore any Nash equilibrium in `BLW` is also a Nash equilibrium in `PROSPER`. When losers bid their true value, by Lemma 3.4, we can restrict ourselves to the Nash equilibria of `PROSPER` of the form $(0, \ldots, 0, b_k, r_{k+1}, \ldots, r_n)$, and the same reasoning as above can be used to show the converse. $\square$

C. PROOFS IN SECTION 6

C.1 Proof of Theorem 6.1

Proof. We start by showing that the final price p is no less than r_α. Assume by contradiction that $p < r_\alpha$. Then, as lenders never bid below their own interest rate, all winning lenders have an interest rate less than r_α, hence belong to the set $\{L_1, \ldots, L_{\alpha-1}\}$. However, since $\sum_{i=1}^{\alpha-1} a_i < D$, the borrower's demand is not fulfilled by the winners of the auction, which is impossible as the total budget of lenders exceeds the demand. Hence $p \geq r_\alpha$.

We now show that $p \leq r_\beta$. Assume by contraction that $p > r_\beta$. Then, all lenders with a true interest no greater than r_β win the auction. Since the total budget of those lenders is greater than D, at least one lender has a budget that is not exhausted. The Prosper mechanism allows exactly one winner to have a non-exhausted budget, let L_j be this lender. Then $L_1, \ldots, L_{j-1}, L_{j+1}, \ldots, L_\beta$ are all winners and exhaust their entire budget. Note that $L_j \in \Delta$ since the budget of those lenders is at least D. We remarked in § 3 that β is the largest index of a VCG winner when the set of lenders is $\{L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n\}$ for any L_i. Therefore the total budget of the winners who completely exhaust their budget, $L_1, \ldots, L_{j-1}, L_{j+1}, \ldots, L_\beta$, is at least D. This contradicts the fact that L_j is allocated a positive amount. Hence $p \leq r_\beta$. $\square$

C.2 Proof of Theorem 6.2

Proof. Let p be the final price. By the above lemma, we know that $p \geq r_\alpha$.

All lenders in Δ are winners of `PROSPER DYNAMICS`. If there is $L_i \notin \Delta$ such that $x_i^{(T)} > 0$, then $p \geq r_{\alpha+1}$ and there is $L_j \in \Delta$ who does not exhaust her budget. This is, however, impossible as myopic greedy behavior would lead L_j to decrease her bid whenever receiving a partial allocation. Therefore the set of winners in `PROSPER DYNAMICS` with myopic greedy lenders is Δ.

For the same reason, no VCG winner whose budget is not exhausted should have a final bid higher than r_α. Hence, if the total budget of VCG winners exceeds D, the final price is r_α. If all VCG winners exhaust their budget, *i.e.* $\sum_{L_i \in \Delta} a_i = D$, then for any price above $r_{\alpha+1}$, lender $L_{\alpha+1}$ can bid just below the price and get a positive allocation, therefore the final bid of lender $L_{\alpha+1}$ is $r_{\alpha+1}$. Since no lender bid below their own interest rate, $r_{\alpha+1}$ is the lowest losing bid and is the final price. $\square$

C.3 Proof of Theorem 6.3

Proof. Let $L_j \in \Delta$ be the lender in Δ whose VCG payment is affected by L_β. We remark that L_j must be a winner, since all VCG winners belong to the set of winners in the dynamic process. If the final price $p < r_\beta - \epsilon$, then there is a previous round where L_j bids $r_\beta - \epsilon$. We show that, when that is the case, L_j always get a positive allocation, so that L_j is not willing to lower her bid, which contradicts to $p < r_\beta - \epsilon$. Indeed, when L_j bids $r_\beta - \epsilon$, lenders who bids less than or equal to L_j belong to $S_j = \{L_1, \ldots, L_{\beta-1}\} \backslash \{L_j\}$. Since β is the index of the largest VCG winner when considering the set of lenders $\{L_1, \ldots, L_{j-1}, L_{j+1}, \ldots, L_n\}$, the total budget of the lenders of S_j is less than D. This implies that L_j must receive a positive allocation for that round. $\square$