Charity Auctions on Social Networks

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Abstract

Charitable giving is influenced by many social, psychological, and economic factors. One common way to encourage individuals to donate to charities is by offering to match their contribution (often by their employer or by the government). Conitzer and Sandholm introduced the idea of using auctions to allow individuals to offer to match the contribution of others. We explore this idea in a social network setting, where individuals care about the contribution of their neighbors, and are allowed to specify contributions that are conditional on the contribution of their neighbors.

We give a mechanism for this setting that raises the largest individually rational contributions given the conditional bids, and analyze the equilibria of this mechanism in the case of linear utilities. We show that if the social network is strongly connected, the mechanism always has an equilibrium that raises the maximum total contribution (which is the contribution computed according to the true utilities); in other words, the price of stability of the game defined by this mechanism is one. Interestingly, although the mechanism is *not* dominant strategy truthful (and in fact, truthful reporting need not even be a Nash equilibrium of this game), this result shows that the mechanism always has a full-information equilibrium which achieves the same outcome as in the truthful scenario. Of course, there exist cases where the maximum total contribution even with true utilities is zero: we show that the existence of non-zero equilibria can be characterized exactly in terms of the largest eigenvalue of the utility matrix associated with the social network.

1 Introduction

Charitable giving constitutes an essential and growing part of the world economy, providing benefits such as helping people overcome natural disasters, reducing income inequality, and funding public projects. According to the Giving USA survey [13], charitable giving in the United States totaled over $295 billions in year 2006, over 2.7% of the total personal income. The vast majority of this sum is donated by individuals, and only a small portion (1.3%) is from mega-gifts that receive media attention.

Reasons why individuals donate to charities, and ways to encourage such donations, have been the subject of extensive research; see [1, 19] for surveys. These reasons include social, psychological, and economic factors. One of the common techniques governments or employers use to encourage donations from individuals is by increasing the benefit of a donation through gift-matching programs. Experimental and field studies have shown that offering to match the contribution of a donor in fact has a significant effect on the likelihood and the amount of a donation [10, 15].

Conitzer and Sandholm [6] introduced the idea of charity auctions in which each bidder declares how much she is willing to contribute as a function of the total contribution from all donors. This allows individual donors to offer to match the contribution of others, providing them with the same added incentive as a government or employer-sponsored matching-gift program. Conitzer and Sandholm consider the problem in its most general form with donations to multiple charities, where the auction mechanism acts as a platform for negotiation among bidders about how much to give as well as to which charities. In this setting, they prove strong hardness results on the clearing problem the auction mechanism needs to solve. As we will observe later in this paper, these hardness results are a consequence of the combinatorial nature of the problem with multiple charities, and do not apply to the case of a single charity campaign, which is the focus of this paper.

An implicit assumption in the model considered by Conitzer and Sandholm is that individuals only care about the total amount donated to the charity, and not on where this donation comes from. However, empirical evidence suggests that charitable giving is highly influenced by contributions from one's social circle (see, for example, [5]). In fact, the Giving USA survey [12] finds that the most important reason that people give for making contributions is being asked to give by someone they know well. Motivated by these observations, we study charity auctions among bidders on a social network, where each bidder can condition his

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or her contribution on the contribution of her neighbors.

We observe that in this setting (which will be defined in further detail in Section 2) there is a unique maximal feasible payment vector. This follows from the fact that feasible payment vectors correspond to fixed points of an increasing function on a lattice. This results in a natural clearing mechanism for the auction, which, as we will observe, can be computed efficiently. The main technical contribution of our paper is the analysis of full-information equilibria of this mechanism, assuming the agents have linear utility functions and a budget (which we will argue is a reasonable assumption).

We show that this mechanism always has an equilibrium which results in the optimal outcome, assuming that the underlying social network is strongly connected. Here, by optimal we mean the outcome that yields the highest total contribution subject to individual rationality constraints. In other words, the price of stability of the game defined by this mechanism is one.[1] It is worth noting that this mechanism is *not* dominant strategy truthful, and in fact, truthful reporting need not even be a Nash equilibrium for this game, yet our result shows that the mechanism always has a full-information equilibrium which achieves the same outcome as in the truthful scenario.

A natural question is how the topology of the underlying social network impacts the success of a fundraising campaign. Using our result about the existence of optimal equilibria, we give a characterization of utility profiles that admit a non-zero equilibrium. Our characterization is in terms of the largest eigenvalue of the utility matrix associated with the social network- we show that non-zero contributions can be raised in equilibrium if and only if the largest eigenvalue of this matrix is at least 1.

Related work. The closest paper in the literature to our work is that of Conitzer and Sandholm [6], which defines the notion of a charity auction and studies the computational complexity of the clearing problem for such auctions. There has been plenty of work, both theoretical and experimental, on charity in the economics literature, see for example the survey papers by Andreoni [1] and Vesterlund [19]. From the mechanism design standpoint, the relevant literature is that on the private provision of public goods. In particular, Guttman [8] and Varian [17, 18] study mechanisms that are based on a notion of *subsidy.* Bagnoli and Lipman [4] gives a simple social-welfare maximizing mechanism for provision of public goods. However, the models and the mechanisms considered in these paper are different from ours. In particular, none of these papers consider a social network setting.

There have been a number of relevant field and laboratory experiments as well. Karlan and List [10] report a field experiment examining the effect of matching funds on charitable contributions, and find that matching contributions increase both the likelihood and amount of a donation. Carmen [5] studies the role of social influence in the workplace on the choice of charitable organizations to receive one's donations, and finds that the designation of an individual gift is affected by social influences. Rege and Telle [14] and Andreoni and Petrie [3] show in laboratory studies that revealing the identities of givers and amounts contributed can have positive impact on donations; similar social effects are shown in a field experiment by Soetevent [16]. Eckel and Grossman [7] compare the effect of rebates and matching mechanisms in a fundraising campaign and show that matching mechanisms result in larger total contributions.

2 The model

In its most general form, the setting of a single-charity fundraising campaign can be described as follows: each of the n participating agents has a utility function u_i for donations to charity. This function maps a vector $\mathbf{x}$ in $\mathbb{R}^n$, where x is the profile of contributions from all agents (including i), to a number $u_i(\mathbf{x})$ indicating how much agent i values this profile. We assume that u_i is a non-decreasing function of x_j for every $j \neq i$, and a non-increasing function of x_i.

Throughout most of this paper, our focus is on the case where the utility function of each agent is a linear function of the contributions from the neighbors of this agent in an underlying social network, subject to a hard budget constraint. More precisely, we assume there is a directed graph G with agents as its vertices. An agent's type is specified by a non-negative row vector $A_i = (a_{ij})$ where $a_{ij} > 0$ if and only if $(i, j) \in E(G)$, and a budget B_i. The overall utility of agent i is given by

$$u_i(\mathbf{x}) = \begin{cases} \sum_j a_{ij} x_j - x_i & \text{if } x_i \leq B_i \\ -\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

A special case of this model which is easier to understand and captures the main properties of the model is when utility functions are a function of the total contribution from neighbors in the social network (*i.e.*, not weighted differently for different neighbors): in this case, the type of each agent i is specified by two non-negative numbers a_i and B_i. The value of agent i for a

[1] Note that in all interesting cases, this game has an equilibrium in which everybody donates 0. Therefore, one cannot hope to prove a similar statement for the worst equilibrium of the game.

contribution profile $\mathbf{x} \in \mathbb{R}^n$ (not counting the disutility caused by having to pay x_i) is given by $a_i \sum_{j \in N(i)} x_j$, where $N(i)$ denotes the set of nodes j that have an edge from i ($(i,j) \in E(G)$). The overall utility of this agent can be written as

$$u_i(\mathbf{x}) = \begin{cases} a_i \sum_{j \in N(i)} x_j - x_i & \text{if } x_i \leq B_i \\ -\infty & \text{otherwise.} \end{cases}$$

Some of our results hold even for more general utility functions and budget constraints (an important exception, which will be noted later, is the result of Section 4 which, as we will observe, does not hold if utility functions are non-linear).

Before defining the notion of a charity auction, we elaborate on the three assumptions we are making in this model:

- **Utilities are linear.** In general, one might expect the utility function of an agent to be a concave function of the contribution of others, since the marginal benefit of additional donations should eventually decrease as the amount of collected funds goes to infinity. However, in most fundraising campaigns, the donation of each individual (or a small group of individuals) is quite small compared to the total amount collected. Therefore, it is reasonable to assume that the marginal benefit of additional donations is almost constant in the relevant range. Furthermore, from a practical point of view, having to specify a non-linear utility function increases the cognitive burden of bidding in the auction, making it hard for non-sophisticated users to participate in the auction, while linear utility functions correspond to the familiar notion of gift-matching.[2]

- **Donations are limited by budget constraints.** Clearly, individuals are constrained by their budget when deciding how much to donate to each charity. One might argue that these budget constraints are often flexible to some degree. A more general model, used by Conitzer and Sandholm [6], considers a *donation willingness* function $w : \mathbb{R} \mapsto \mathbb{R}$, which maps the utility the agent receives from the contribution profile to the maximum amount she is willing to donate. However, specifying such a function seems to be too difficult to be practical. Furthermore, without any constraint on the function w, the utility loses its meaning, and therefore it is impossible to prove any general result on the equilibrium properties of the mechanism.

- **Single charity campaign.** We assume in our model that the fundraising campaign concerns only one charity (or a set of charities that are all equally acceptable from the agents' point of view). Again, the first reason for this assumption is to make it easier for the agents to participate in the campaign. Also, the results of Conitzer and Sandholm [6] show that if multiple charities are considered, most problems become computationally intractable, even ignoring the social network aspect of the problem.

We now formally define a charity auction.

Charity auctions. A charity auction is a mechanism that elicits from each agent her private type (A_i, B_i), and based on these values, outputs a contribution profile. More generally, the mechanism asks each bidder i for a *willingness-to-pay* function $v_i : \mathbb{R}^n \mapsto \mathbb{R}$, which specifies the maximum amount this agent is willing to pay at each contribution profile; in other words, by bidding v_i, bidder i declares a contribution profile $\mathbf{x}$ as acceptable from her point of view if $x_i \leq v_i(\mathbf{x})$.[3] In the case of linear utilities with budgets, we can take $v_i(\mathbf{x}) = \min(B_i, \sum_j a_{ij} x_j)$. It is easy to see that since the utility function $u_i(\mathbf{x})$ was assumed to be a *non-decreasing* function of x_j ($j \neq i$), we can restrict v_i to be a non-decreasing function in all x_j's. Furthermore, we assume that each v_i is a *bounded* function.[4]

A *feasible* outcome is a vector $\mathbf{x} \in \mathbb{R}^n$ such that for each i, $x_i \leq v_i(\mathbf{x})$. A mechanism satisfies *individual rationality* if it always outputs a feasible contribution profile.

A special case of our model, which will receive special treatment, is when the graph G is complete (i.e., for every i and j, $(i,j) \in E(G)$). Note that this is also a special case of the model considered by Conitzer and Sandholm [6], since in this case the utilities only depend

[2] A web-based system based on the model described in this section is currently implemented and will be made publicly available soon. In this system, a user can start a new campaign, participate in an existing campaign by specifying her budget and the amount of matching funds she is willing to provide for every dollar contributed by her friends, or invite new friends to join the campaign.

[3] Note that while conceptually it is more natural to restrict the willingness-to-pay function v_i to be a function of the contribution profile $\mathbf{x}_{-i}$ of other agents, for the sake of simplicity we allow v_i to depend on x_i as well. This only simplifies the math, and does not change the expressiveness of the bidding language. For example, in the case of linear utilities (with $a_{ii} < 1$), taking $v_i(\mathbf{x}) = \min(B_i, \sum_j a_{ij} x_j)$ and $v_i(\mathbf{x}) = \min(B_i, \frac{1}{1-a_{ii}} \sum_{j \neq i} a_{ij} x_j)$ both lead to the same set of acceptable profiles from the point of view of i.

[4] However, we do not assume that the auctioneer knows the bound on v_i's.

on the total donation to the charity. In Section 4 we first prove our result in this special case, since in this case exact equilibrium conditions can be written explicitly, providing intuition for the proof of the more general case.

3 The mechanism

Recall that a charity auction mechanism needs to compute a feasible payment vector $\mathbf{x}$ given non-decreasing functions $v_i : \mathbb{R}^n \mapsto \mathbb{R}$ (where $v_i(\mathbf{x}) = \min(B_i, \sum_j a_{ij}x_j)$ in our model). Moreover, since often there are many feasible payment vectors (e.g., $\overline{0}$ is always feasible), the mechanism must output a feasible payment vector that maximizes some natural objective function, such as the total contribution $\sum_i x_i$. The following lemma shows that there is a unique maximal feasible payment vector. This vector clearly maximizes the total contribution and any other non-negative objective function. In the following lemma, for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\max(\mathbf{x}, \mathbf{y})$ denotes a vector in $\mathbb{R}^n$ whose i'th coordinate is $\max(x_i, y_i)$. Also, by $\mathbf{x} \leq \mathbf{y}$ we mean $x_i \leq y_i$ for every i.

LEMMA 3.1. *For every two feasible payment vectors $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, $\max(\mathbf{x}, \mathbf{x}')$ is also a feasible payment vector. Therefore, there is a unique feasible payment vector $\mathbf{x}^*$ such that $\mathbf{x} \leq \mathbf{x}^*$ for every feasible payment vector $\mathbf{x}$.*

Proof. Let $\mathbf{y} = \max(\mathbf{x}, \mathbf{x}')$ for two feasible payment vectors $\mathbf{x}$ and $\mathbf{x}'$. By feasibility of $\mathbf{x}, \mathbf{x}'$, we have $x_i \leq v_i(\mathbf{x})$ and $x'_i \leq v_i(\mathbf{x}')$ for every i. Since v_i is non-decreasing and $\mathbf{x} \leq \mathbf{y}$, we obtain $x_i \leq v_i(\mathbf{x}) \leq v_i(\mathbf{y})$. Similarly, $x'_i \leq v_i(\mathbf{y})$. Hence, $y_i = \max(x_i, x'_i) \leq v_i(\mathbf{y})$.

Note that the above proof works not only for the maximum of two feasible vectors, but also for the supremum of any set of feasible vectors. Therefore, if we let $\mathbf{x}^*$ denote the supremum of all feasible vectors (this vector is well defined since v_i's are bounded and therefore the set of feasible vectors is a bounded subset of $\mathbb{R}^n$), by this argument $\mathbf{x}^*$ is a feasible payment vector, and by definition, $\mathbf{x} \leq \mathbf{x}^*$ for every feasible payment vector $\mathbf{x}$. $\square$

An alternative way to prove the above lemma is to note that the maximal feasible point $\mathbf{x}^*$ corresponds to the maximal fixed point of a non-decreasing function on the lattice $\mathbb{R}^n$, and then use Tarski's fixed point theorem to show that such a fixed point always exists and is unique. We will use this interpretation later in the paper (in the proof of Theorem 4.2) in order to apply comparative statistics theorems on Tarski fixed points.

Given the above result, the mechanism can simply elicit v_i's and output the unique maximal feasible vector with respect to these functions.

Computation of the outcome. When utility functions are linear, the maximal feasible vector can be computed in polynomial time by solving the following linear program.

$$\begin{array}{ll} \text{maximize} & \sum_i x_i \\ \text{subject to} & \forall i: \quad x_i \leq \sum_j a_{ij}x_j \\ & \forall i: \quad x_i \leq B_i \end{array}$$

An alternative approach for computing the maximal feasible vector is to start from $\mathbf{x}^{(0)} = (B_1, \ldots, B_n)$, and iteratively define $x_i^{(k)} = \min(x_i^{(k-1)}, v_i(\mathbf{x}^{(k-1)}))$ for every i, k. This algorithm always converges to the solution $\mathbf{x}^*$, as long as the functions v_i are continuous. However, this procedure might take infinite time to converge. In the case of linear utilities, this algorithm can be made more efficient as follows: in every step k, let $S_k = \{i : \sum_j a_{ij}x_j^{(k-1)} < B_i\}$, and solve a system of linear equations to find $\mathbf{x}^{(k)}$ satisfying $x_i^{(k)} = B_i$ for all $i \notin S_k$ and $x_i^{(k)} = \sum_j a_{ij}x_j^{(k)}$ for all $i \in S_k$. It is straightforward to show that in each iteration, if $x^{(k)} \neq x^{(k-1)}$, the size of S_k increases by at least one. Therefore, the algorithm computes the solution after solving at most n systems of linear equations.

4 Equilibrium analysis

In this section, we consider the strategic behavior of the agents in a charity auction. The mechanism defined in the previous section induces the following game among the bidders: each agent reports a tuple (A'_i, B'_i) as her type. The outcome of the charity auction is the vector of payments $\mathbf{x} = P(A', B')$ as defined in the previous section. Given this outcome, agent i derives a utility of $\sum_j a_{ij}x_j - x_i$ if $x_i \leq B_i$ and $-\infty$ otherwise. We are interested in studying full-information equilibria of this game, i.e., a set of bids (A', B') such that even knowing other agents' bids, no agent has an incentive to change her bid. The main result of this section is that this game always has an equilibrium which yields the same payment vector $\mathbf{x}$ as the payment vector $P(A, B)$ with respect to the true types. It is easy to see that this is the highest total contribution one might hope to achieve subject to individual rationality. For this reason, we call this outcome the *optimal* outcome. Note that the optimal outcome is not necessarily the same as the social-welfare maximizing outcome, if we define the social welfare as the sum of the utilities of the bidders[5]. Since our model is motivated by charities that use the donations to improve the well-being of others (not just the contributors), the total contribution is a more reasonable objective function than the sum of the utilities of the bidders.

We first study this problem in the special case where the graph G is complete (Section 4.1). In this case, the utility of agent i is $a_i X - x_i$ if $x_i \leq B_i$, where $X := \sum_j x_j$ is the total contribution of all agents. We give a characterization of the equilibria in this special case, providing the intuition for the proof of the more general case, which will be presented in Section 4.2.

4.1 The case of complete graph

Let H denote the game defined by our charity mechanism where the type of agent i is (a_i, B_i). Consider a full-information equilibrium of this game, and let (a'_i, B'_i) denote the bid of agent i at this equilibrium and $x = P(a', B')$ be the resulting payment vector. In this section, we give conditions that characterize such equilibria and the associated payment vector. The first step is to prove that for each agent i, in order to decide if (a'_i, B'_i) is a best-response to the strategy of other agents, it is enough to only look at the final payment x_i of this agent. This is an important step in the proof, and also in the proof of the general case: analyzing equilibria of mechanisms in which the type of each agent is multi-dimensional is known to be notoriously difficult; in our setting, while the type of each agent is multi-dimensional, this lemma allows us to focus on only one relevant parameter, x_i.

To state the lemma, we define a modified game H' in which agents $j \neq i$ submit bids (a'_j, B'_j) as before, but agent i has to submit a value z. The mechanism then computes the payment vector x using the mechanism described in Section 3, but with the functions v_i defined as the constant function $v_i(x) = z$ (other v_j's are defined as before: $v_j(x) = \min(B'_j, a'_j X)$). In other words, in this game agent i reports her *unconditional payment* z. Let $P(z, a'_{-i}, B'_{-i})$ denote the outcome of this game.

LEMMA 4.1. *Let (a', B') be a set of bids, and denote $x = P(a', B')$. Then (a'_i, B'_i) is a best-response to the strategy profile (a'_{-i}, B'_{-i}) in the game H if and only if x_i is the best-response to this strategy profile in the game H'.*

Proof Sketch. By the definition of the mechanism in the previous section, it is not hard to see that $P(x_i, a'_{-i}, B'_{-i}) = x$. Let z denote i's best-response to the strategy profile (a'_{-i}, B'_{-i}) in the game H', and let $y := P(z, a'_{-i}, B'_{-i})$. Note that $z = y_i \leq B_i$. We define $a''_i := y_i / Y$, where $Y = \sum_j y_j$. It is straightforward to show that $y = P((a''_i, B_i), a'_{-i}, B'_{-i})$, i.e., if i bids (a''_i, B_i) in the game H, the outcome of the game will be y. This means that if i's best-response z in the game H' is different from x_i, then there is also a strategy (a''_i, B_i) in the game H which performs better than (a'_i, B_i), and hence (a'_i, B_i) cannot be a best response. The other direction is proved similarly. □

Next, we give necessary and sufficient condition for an unconditional payment to be the best-response in the game H'.

LEMMA 4.2. *The strategy x_i is the best-response to the strategy profile (a'_{-i}, B'_{-i}) in the game H' if the following conditions hold:*

$$
\begin{aligned}
x_i = B_i \quad &\Rightarrow \quad a_i + \sum_{j \in S^+ \setminus i} a'_j \geq 1, \\
x_i < B_i \quad &\Rightarrow \quad a_i + \sum_{j \in S^+ \setminus i} a'_j \geq 1 \quad \textit{and} \\
&\qquad\; a_i + \sum_{j \in S^- \setminus i} a'_j \leq 1,
\end{aligned}
$$

where $x = P(x_i, a'_{-i}, B'_{-i})$, $X = \sum_j x_j$, $S^- = \{j : a'_j X < B'_j\}$, and $S^+ = \{j : a'_j X \leq B'_j\}$.

Proof. We need to argue that with the above conditions, bidder i cannot benefit by changing her bid from x_i. If $x_i = B_i$, then i can only decrease his bid. Consider the utility function $u_i = a_i X - x_i$ of i only as a function of x_i, holding the values of (a'_{-i}, B'_{-i}) constant. We compute the left derivative of this function at x_i. Let $C = x_i + \sum_{j \notin S^+} B'_j$. Then, since $x = P(x_i, a'_{-i}, B'_{-i})$, for every $j \in S^+ \setminus i$ we have

$$x_j = a'_j \cdot (\sum_{k \in S^+ \setminus i} x_k + C).$$

Summing over all $j \in S^+ \setminus i$ and denoting $x(S^+ \setminus i) := \sum_{j \in S^+ \setminus i} x_j$ and $a'(S^+ \setminus i) := \sum_{j \in S^+ \setminus i} a'_j$, we obtain

$$x(S^+ \setminus i) = a'(S^+ \setminus i) \cdot (x(S^+ \setminus i) + C)$$

Therefore,

$$
\begin{aligned}
X &= x(S^+ \setminus i) + C \\
&= \frac{C}{1 - a'(S^+ \setminus i)} \\
&= \frac{x_i + \sum_{j \notin S^+} B'_j}{1 - a'(S^+ \setminus i)}.
\end{aligned}
$$

[5] An exception is the case of a complete graph. It is not hard to prove that in this case the optimal outcome coincides with the social-welfare maximizing outcome.

The above equation gives the value of X as a function of x_i. It is easy to see that the above equation holds not only at x_i, but also on a left neighborhood of x_i. Therefore, the left derivative of X with respect to x_i is $1/(1-a'(S^+ \setminus i))$. Thus, the left derivative of the utility function $u_i = a_i X - x_i$ of agent i with respect to x_i is $a_i/(1-a'(S^+ \setminus i)) - 1$. To ensure that agent i cannot benefit by decreasing her bid, this left derivative must be nonnegative:

$$\frac{a_i}{1-a'(S^+ \setminus i)} - 1 \geq 0,$$

which we can rearrange to get the first condition. Note that we only need to consider the slope at x_i, since u_i is a piecewise linear concave function, and the slope at lower x_i can only increase (since the set S^+ can only grow larger when payments decrease).

A bidder with $x_i < B_i$ can either increase or decrease his payment; the condition to have no incentive to decrease his bid is that the left derivative of the utility function at x_i is nonnegative, which is the same as above. Using a similar argument, such an i will not increase his bid if the right derivative of the utility at x_i is non-positive; this works out to

$$\frac{a_i}{1-a'(S^- \setminus \{i\})} - 1 \leq 0,$$

where $a'(S^- \setminus \{i\}) = \sum_{j \in S^- \setminus i} a'_j$. Again, as x_i increases, S^- shrinks, and the slope can only decrease, so it is sufficient to check the right derivative at x_i. Rearranging, we get the second condition. □

Now we use these two lemmas to characterize the set of payment vectors which can be outcomes in a Nash equilibrium of the charity auction.

THEOREM 4.1. *Consider the game H with agent types (a, B), and let $\mathbf{x} \in \mathbb{R}^n$ and $X = \sum_j x_j$. There exists a full-information Nash equilibrium of the this game with the outcome x if the following conditions hold.*

$$\forall\ i : \frac{x_i}{X} \leq a_i, \tag{4.2}$$

$$\forall\ i \text{ s.t. } x_i < B_i : \sum_{\substack{j:j \neq i \\ x_j < B_j}} \frac{x_j}{X} \leq 1 - a_i, \tag{4.3}$$

$$\exists\ i \text{ s.t. } x_i = B_i. \tag{4.4}$$

Proof. Let $a'_i = x_i/X$ for all i. We claim that if the above conditions are satisfied, (a', B) constitute a Nash equilibrium of H with the outcome x. First we show that $x = P(a', B)$. By the definition of a'_i's, x is a feasible payment vector for (a', B); we need to show it is a maximum feasible solution. Let $y = P(a', B)$ be the maximum feasible solution and assume $Y = \sum_{i=1}^n y_i > X$. By the feasibility of y, for every i we have $y_i \leq a'Y = x_i Y/X$. Furthermore, by the above conditions there exists at least one i such that $x_i = B_i$. For this i, we have $y_i \leq B_i < x_i Y/X$. Therefore,

$$Y \quad < \quad \sum_{i=1}^n \frac{x_i}{X} Y = Y,$$

which is a contradiction. Therefore x must be the maximum feasible outcome at (a', B).

Next, we use Lemma 4.2 to show that for every i, x_i is the optimal unconditional payment given (a'_{-i}, B_{-i}), and therefore (a', B) is an equilibrium. By the definition of a'_i's, S^+ is the set of all bidders, and $S^- = \{j : x_j < B_j\}$. Since $a'_j = x_j/X$, $\sum_{j=1}^n a'_j = 1$, and the first condition on the bids reduces to

$$a'_i = \frac{x_i}{X} \leq a_i$$

for all i, which is true by condition (4.2). For i such that $x_i < B_i$, the second condition for not having an incentive to deviate from his payment reduces to

$$\sum_{j \in S^- \setminus i} \frac{x_j}{X} \leq 1 - a_i,$$

which is true by condition (4.3). This shows that the strategy profile (a', B) is in fact a Nash equilibrium of the game. □

COROLLARY 4.1. *There exists a set of bids (a', B') which are a Nash equilibrium, and achieve the optimal outcome $\mathbf{x} = P(a, B)$.*

Proof. Conditions (4.2) and (4.4) in Theorem 4.1 are satisfied since $\mathbf{x} = P(a, B)$ is the maximum feasible payment. To verify condition (4.3), let $X^- := \sum_{j:x_j < B_j} x_j$. We have $x_i = a_i X$ for every i such that $x_i < B_i$. Summing up these equations, we obtain

$$X^- = (\sum_{i:x_i < B_i} a_i)(X^- + \sum_{i:x_i = B_i} B_i),$$

Since there is at least one i with $x_i = B_i$, the above equation cannot be satisfied unless

$$\sum_{i:x_i < B_i} a_i < 1.$$

Since $\frac{x_j}{X} \leq a_j$ for all j, the above inequality implies

$$a_i + \sum_{j:j \neq i, x_j < B_j} \frac{x_j}{X} < 1$$

for every i. Therefore, condition (4.3) is satisfied. $\square$

The above corollary shows that our mechanism has an equilibrium which extracts the maximum possible revenue of all individually rational mechanisms: the outcome $\mathbf{x}$ of any individually rational mechanism must satisfy $u_i(\mathbf{x}) \geq 0$, *i.e.*, $x_i \leq a_i X$, and $x_i \leq B_i$. Therefore, any such $\mathbf{x}$ will be a feasible vector with respect to the true (a, B), and hence, by definition, $\mathbf{x} \leq P(a, B)$.

4.2 General networks with linear utilities

Now we consider general linear utility functions, *i.e.*, when the underlying graph on the set of bidders need not be a complete graph. In this case, a bidder's bid is a row vector $A'_i = (a'_{i1}, a'_{i2}, \ldots, a'_{in})$ and a budget B'_i, which are used to calculate his payments. We investigate the question of whether the optimal outcome (*i.e.*, the payments computed from the true values) can be supported in an equilibrium. In other words, we would like to know whether for every utility profile (A, B), there exits bids (A', B') that form a Nash equilibrium and satisfy $P(A', B') = P(A, B)$.

Before we state our results, we state the following theorem about non-negative matrices, which we will use in our proofs.

THEOREM A. ([9],THEOREM 8.3.3) *Let A be a non-negative $n \times n$ matrix. The eigenvalue of A with the largest magnitude, $\lambda_{\max}$, is real, and there is a non-negative vector $x \geq 0$, $x \neq 0$, such that $Ax = \lambda_{\max} x$. Further,*

$$\lambda_{\max} = \max_{x \geq 0, x \neq 0} \min_{1 \leq i \leq n, x_i \neq 0} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j.$$

The eigenvalue $\lambda_{\max}$ is referred to as the Perron-Frobenius eigenvalue of the non-negative matrix A.

THEOREM 4.2. *The optimal outcome $\mathbf{x} = P(A, B)$ can be supported in an equilibrium if the underlying graph of A is strongly connected.*

Proof Sketch. The payment of bidder i in the optimal outcome satisfy $x_i = \min(A_i^T \mathbf{x}, B_i)$. Define the matrix A' as

$$a'_{ij} = a_{ij} \frac{x_i}{\sum_{j \in N(i)} a_{ij} x_j},$$

where $\mathbf{x}$ is the payment vector according to the true utilities. We claim that the strategy profile (A', B) is an equilibrium of the game, and the outcome of the mechanism at this strategy profile is $\mathbf{x}$.

First we show that $P(A', B) = \mathbf{x}$. Clearly $\mathbf{x}$ is a feasible payment vector with respect to (A', B), so $\mathbf{x} \leq P(A', B)$; we only need to show it is the maximum feasible vector. To show this, we use the observation that the outcome $P(A', B)$ of the mechanism is the maximal fixed point of the function v that maps any vector $y \in \mathbb{R}^n$ to the vector $(v_1(y), \ldots, v_n(y))$, where $v_i(y) = \min(B_i, {A'_i}^T y)$. This function is an isotone function on the lattice $\mathbb{R}^n$, and it is not hard to show that its largest fixed point corresponds to the outcome $P(A', B)$. Furthermore, note that the function v_i is an increasing function of each a'_{ij}. Thus, by the comparative statics theorem of [11], increasing the value of a'_{ij} increases the value of $P(A', B)$. Therefore, since for every i and j, $a'_{ij} \leq a_{ij}$, we have $P(A', B) \leq P(A, B) = \mathbf{x}$.

Next, we prove that (A', B) is an equilibrium. An argument similar to the one in Lemma 4.1 shows that a bidder k has no incentive to unilaterally deviate from his bid (A'_k, B_k) if the corresponding payment x_k is his optimal unconditional payment given other bidders' bids.

Now consider a particular bidder k. First note that this bidder has no incentive to increase his unconditional payment, since x_k is the maximum individually rational payment (no larger payment satisfies $x_k \leq \min(A_k^T \mathbf{x}, B_i)$, since $\mathbf{x}$ is computed using the true utilities A by P). Now we show that there is no incentive to decrease the unconditional payments. Permute and partition the matrix A' as

$$A' = \begin{bmatrix} {A'}^{+} & b'_k \\ {a'_k}^T & a'_{kk} \end{bmatrix},$$

where ${A'}^{+}$ is the $n-1 \times n-1$ submatrix of A' removing the kth row and column, and b'_k is the kth column of A'. We will show in Lemma 4.3 that the matrix $I - {A'}^{+}$ is invertible when G is strongly connected.

Consider an arbitrary unconditional payment $y_k \leq x_k$ by bidder k, and let $\mathbf{y}_{-k}$ be the vector of payments of all other agents computed according to ${A'}^{+}$. We want to find the condition under which y_k is a best response to ${A'}^{+}$. For all bidders $j \neq k$ we have $A'_j \mathbf{x} = x_j \leq B_j$. Therefore, all such bidders must satisfy $A'_j \mathbf{y} \leq B_j$ and hence $y_j = A'_j \mathbf{y}$. Therefore, the payments of bidders other than k for all $y_k \leq x_k$ can be calculated as

$$\mathbf{y}_{-k} = {A'}^{+} \mathbf{y}_{-k} + y_k b'_k \Rightarrow \mathbf{y}_{-k} = y_k (I - {A'}^{+})^{-1} b'_k. \tag{4.5}$$

The utility of bidder k is

$$\sum_{j \in N(k)} a_{kj} y_j - y_k,$$

where a_{kj}s are entries from the true matrix A.

The condition for bidder k to not decrease his payment is that the left-derivative of his utility at x_k is nonnegative, which we can write using (4.5) as

$$a_k^T(I - A'^+)^{-1}b'_k + a_{kk} - 1 \geq 0.$$

Since $(I - A'^+)^{-1}b'_k = (1/x_k)\mathbf{x}_{-k}$, the condition under which x_k is the optimal payment given the remaining bids is

$$a_k^T\mathbf{x}_{-k} + a_{kk}x_k \geq x_k. \tag{4.6}$$

Since the payment vector $\mathbf{x}$ is feasible with respect to (A', B), we have

$$x_k \leq {a'_k}^T\mathbf{x}_{-k} + a'_{kk}x_k.$$

This, combined with the inequality $a'_{ij} \leq a_{ij}$ for every i, j, implies (4.6), completing the proof of the theorem. □

The only remaining step is the following lemma.

Lemma 4.3. *The matrix $I - A'^+$ is non-singular if the graph underlying A is strongly connected.*

Proof Sketch. First we show that $\lambda_{\max}(A') = 1$. Let λ be the largest eigenvalue of A', with associated nonnegative (from Theorem A) eigenvector $A'y = \lambda y$. Note that $y_i = 0$ if $x_i = 0$, since if $x_i = 0$, the entire ith row of A' is zero. From the definition of A' and y, we have

$$\frac{y_i}{x_i} = \frac{1}{\lambda}\frac{{a'_i}^T y}{{a'_i}^T x} \leq \frac{1}{\lambda}\max_{j\in N(i)}\frac{y_j}{x_j},$$

which leads to a contradiction for bidder i with $\max y_i/x_i$ if $\lambda > 1$. Since we know that $A'x = x$, 1 is an eigenvalue of A', and so the largest eigenvalue.

Since A'^+ is a principal submatrix of the nonnegative matrix A, $\lambda_{\max}(A'^+) \leq \lambda_{\max}(A') = 1$ ([9], Corollary 8.1.20). If $I - A'^+$ is singular, there exists z such that $A'^+z = z$, where $z \geq 0$, since from Theorem A there is a nonnegative vector corresponding to the largest eigenvalue. Thus z satisfies

$$\begin{aligned} z_i &= \sum_{j\in N(i)\setminus k} z_j \frac{x_i}{\sum_{j\in N(i)} x_j} \\ \Rightarrow \frac{z_i}{x_i} &\leq \frac{\sum_{j\in N(i)\setminus k} z_j}{\sum_{j\in N(i)\setminus k} x_j} \leq \max_{j\in N(i)\setminus k}\frac{z_j}{x_j}, \end{aligned}$$

where the inequality is strict when $k \in N(i)$. Choose bidder i with $\max z_i/x_i$; by the above inequality, all $j \in N(i)$ must have this same value of z_j/x_j, and so on; this leads to a contradiction since the inequality must be strict for a bidder with $k \in N(i)$, and we will reach such a bidder since the graph is strongly connected. So such a z cannot exist, *i.e.*, $I - A'^+$ is non-singular. □

Example. The following simple example shows that the conclusion of Theorem 4.2 (or even Corollary 4.1) need not be true when the utilities of the agents are non-linear. Assume there are two agents, each with a willingness-to-pay function of $2 - 2/(X + 1)$, where $X = x_1 + x_2$ is the total contribution. It is easy to see that with the truthful reporting of these functions, the outcome will be $\mathbf{x} = (1.5, 1.5)$, resulting in a utility of 0 for both agents. However, no matter how much the bid of the second agent is, the first agent can derive a positive utility by contributing, say 0.5. Therefore, the outcome $(1.5, 1.5)$ cannot be supported as an equilibrium.

5 Characterization of successful campaigns

In this section, we characterize utility profiles for which the mechanism succeeds in collecting a non-zero amount from the agents. Recall that the utility of an agent i can be given by the values a_{ij} and B_i. The values a_{ij} constitute the i'th row of a matrix A. Denoting this row by A_i, the utility function of the i'th agent can be written as $u_i(x) = A_i^T.x - x_i$ for $x_i \leq B_i$. Our characterization is in terms of the eigenvalues of the matrix A.

By the result of the previous section, in order to show that the mechanism has an equilibrium with non-zero total contribution, it is enough to show that the payment vector $P(A, B)$ corresponding to agents' true types is non-zero.

Theorem 5.1. *The payment vector $x = P(A, B)$ is non-zero if and only if the matrix A of reported utilities is such that $\lambda_{\max}(A) \geq 1$.*

Proof. The payment vector is the maximum feasible solution of $x_i \leq \min(A_i^T x, B_i)$. First, note that since A is a nonnegative matrix, $\lambda_{\max}$ is real and there exists a nonnegative eigenvector v, from the Perron-Frobenius theorem. Sufficiency is easy: If $Av = \lambda_{\max}v$, with $\lambda_{\max} \geq 1$, then $x = \alpha v$ is a feasible nonzero vector of payments for sufficiently small α (chosen to satisfy the budget constraint $x_i \leq B_i$). So there exists a non-zero feasible payment vector, which means the maximum feasible vector computed by the mechanism is non-zero as well, *i.e.*, has non-zero total contribution.

For necessity, we will use max-min characterization of the Perron-Frobenius eigenvalue from Theorem A:

$$\lambda_{\max}(A) = \max_{x \geq 0, x \neq 0} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i}.$$

Therefore, if $\lambda_{\max} < 1$, there is no nonzero, nonnegative solution to $x \leq Ax$, which means the maximum feasible solution is 0. □

The above theorem implies the following corollary in the case of complete graphs. It is also straightforward to prove this corollary directly.

COROLLARY 5.1. *When utilities are $u_i(\mathbf{x}) = a_i X - x_i$ where X is the sum of all payments, the vector of payments is nonzero if and only if $\sum_{i=1}^{n} a_i \geq 1$.*

Proof. The eigenvalues of A, which has identical columns $(a_1, \ldots, a_n)$, are $\lambda_{\max} = \sum_{i=1}^{n} a_i$, and $\lambda_2 = \ldots = \lambda_n = 0$ repeated $n-1$ times; using the theorem above gives this condition. □

6 Conclusion

In this paper, we defined and studied charity auctions on social networks, and showed that such auctions can achieve an optimal outcome in a full-information Nash equilibrium. We also gave a characterization of campaigns that can raise a non-zero amount in terms of a matrix property of a utility matrix. There are still many interesting open questions related to charity auctions; for example,

- *More than one charity:* One of the assumptions of our model was that the charity campaign concerns only one charity. It would be interesting to generalize this to a setting where multiple charities compete for contributions of the agents.

- *Signaling:* What is the role of the social network and matching contributions in signaling information about the quality of a charity? A related paper is the one by Andreoni [2] which studies the role of leadership giving in signaling information about the quality of a charity.

- *Dynamics:* We proved that our auction mechanism can support the optimal outcome in an equilibrium. However, this mechanism has multiple equilibria, and some of them are far from optimal. It would be interesting to study which of these equilibria will be selected under reasonable assumptions about the dynamics of the game. Can the auctioneer lead the system toward a good equilibrium by announcing or withholding selected pieces of information at various stages of the game?

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References

[1] James Andreoni. Philanthrophy. In *Handbook of Giving, Reciprocity and Altruism*. Elsevier/North-Holland, 2004.

[2] James Andreoni. Leadership giving in charitable fundraising. *Journal of Public Economic Theory*, 8(1):1–22, 2006.

[3] James Andreoni and Ragan Petrie. Public goods experiments without confidentiality: A glimpse into fund-raising. *Journal of Public Economics*, 88:1605–1623, 2004.

[4] Mark Bagnoli and Barton L. Lipman. Provision of public goods: Fully implementing the core through private contributions. *Review of Economic Studies*, 56:583–601, 1989.

[5] Katherine Carman. Social Influences and the Private Provision of Public Goods: Evidence from Charitable Contributions in the Workplace. Stanford Institute for Economic Policy Research Discussion paper No. 02-13, January 2003.

[6] Vincent Conitzer and Tuomas Sandholm. Expressive negotiation over donations to charities. In *Proc. ACM Conference on Electronic Commerce*, May 2004.

[7] Catherine C. Eckel and Philip J. Grossman. Rebate versus matching: does how we subsidize charitable contributions matter? *Journal of Public Economics*, 87:681–701, 2003.

[8] Joel M. Guttman. Understanding collective action: Matching behavior. *American Economic Review*, 68:251–255, 1978.

[9] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, UK, 1985.

[10] Dean Karlan and John A. List. Does price matter in charitable giving? evidence from a large-scale natural field experiment. National Bureau of Economic Research (NBER) Working Papers 12338, 2006.

[11] Paul Milgrom and Chris Shannon. Monotone comparative statics. *Econometrica*, 62:157–180, 1994.

[12] American Association of Fundraising Counsel Trust for Philanthropy. *Giving USA 2002*. American Association of Fundraising Counsel Trust for Philanthropy, 2002.

[13] American Association of Fundraising Counsel Trust for Philanthropy. *Giving USA 2007*. American Association of Fundraising Counsel Trust for Philanthropy, 2007.

[14] Mari Rege and Kjetil Telle. The impact of social approval and framing on cooperation in public good situations. *Journal of Public Economics*, 88:1625–1644, 2004.

[15] Jen Shang and Rachel Croson. Field experiments in charitable contribution: the impact of social influence on the voluntary provision of public goods. Unpublished manuscript, available at `http://opim.wharton.upenn.edu/~crosonr/research/%5B73%5D.pdf`, 2005.

[16] Adriaan R. Soetevent. Anonymity in giving in a natural context – a field experiment in 30 churches. *Journal of Public Economics*, 89:2301–2323, 2005.

[17] Hal Varian. Sequential contributions to public goods. *Journal of Public Economics*, 53(2):165–186, February 1994.

[18] Hal Varian. A solution to the problem of externalities and public goods when agents are well informed. *American Economic Review*, 84:1278–1293, 1994.

[19] Lisa Vesterlund. Why do people give? In *The Nonprofit Sector, 2nd edition*. Yale Press, 2006.